Concentration for Coulomb gases on compact manifolds

I see this article as an excuse to present the inequality below. For any compact Riemannian manifold M of dimension $d \geq 2$, Green function G, heat kernel p_t and volume (probability) measure σ, there exists $C > 0$ such that, for every $(x_1, \ldots, x_n) \in M^n$ and every $t > 0$:

$$\boxed{\frac{1}{n^2} \sum_{i<j} G(x_i, x_j) \geq \frac{1}{2} \int_{M \times M} G\left(\frac{1}{n} \sum_{i=1}^{n} p_t(x_i, \cdot) \mathrm{d}\sigma\right)^{\otimes 2} - t + \frac{1_{d=2}}{8\pi n} \log t - \frac{C}{n t^{d/2-1}}.}$$

This, complemented with the fact that, for the Wasserstein distance W_1,

$$W_1\left(\frac{1}{n} \sum_{i=1}^{n} p_t(x_i, \cdot) \mathrm{d}\sigma, \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}\right) \leq C\sqrt{t},$$

is used to obtain a concentration inequality for Coulomb gases (taking $t = \frac{1}{n^{2/d}}$ to optimize).

Remark 1. *In fact, the proof of the inequality $W_1 \leq C\sqrt{t}$ shows first that $W_2 \leq C\sqrt{t}$, and the concentration inequality can be obtained for the W_2 distance (we should be careful if we want to add a potential).*

Remark 2. *Using the heat kernel is not necessary. Indeed, a similar result holds in the Euclidean case for $p_t(x, \cdot)\mathrm{d}\sigma$ replaced by $\sqrt{t}^d \rho(x + \sqrt{t} y)\mathrm{d}y$ and ρ of finite energy. Nevertheless, in this case, the heat kernel seems like the natural choice since G and p are both obtained from the Laplacian which gives a nice interaction between those objects.*

Idea of the proof

Let $0 = \lambda_0 \leq \lambda_1 \leq \ldots$ be the eigenvalues of minus the Laplacian with orthonormal eigenvectors $(e_n)_{n\geq 0}$. The main (standard) idea is to notice that, as operators,

$$G = \sum_{k=1}^{\infty} \frac{1}{\lambda_n} e_n \otimes e_n^* \quad \text{and} \quad p_t = \sum_{k=0}^{\infty} e^{-t\lambda_n} e_n \otimes e_n^*.$$

Then, as operators, we have

$$G = \int_0^{\infty} (p_s - e_0 \otimes e_0^*)\mathrm{d}s$$

so that

$$p_t \circ G \circ p_t = \int_0^{\infty} (p_{2t+s} - e_0 \otimes e_0^*)\mathrm{d}s = \int_{2t}^{\infty} (p_s - e_0 \otimes e_0^*)\mathrm{d}s.$$

Equivalently, as functions of two variables,

$$\int_{M \times M} G\big(p_t(x, \cdot)\mathrm{d}\sigma\big)\big(p_t(y, \cdot)\mathrm{d}\sigma\big) = \int_{2t}^{\infty} (p_s(x, y) - 1)\mathrm{d}s,$$

which, together with the fact that $p_t(x, y) \geq 0$ and the asymptotic expansion

$$p_t(x, x) = \frac{1}{(4\pi t)^{d/2}} + O\left(\frac{t}{t^{d/2}}\right) \text{ as } t \to 0^+,$$

gives the inequality stated in the box above.