An LDP for empirical measures on Polish spaces

The *main goal* of this article is to provide **simple conditions** for $W_n : M^n \to (-\infty, \infty]$ so that the measures γ_n given by

$$\mathrm{d}\gamma_n = \exp(-n\beta_n W_n)\mathrm{d}\sigma^{\otimes n}$$

satisfy a Laplace principle, i.e.,

$$\frac{1}{n\beta_n} \log \gamma_n(M^n) \text{ converges to } -\inf\{\text{something}\}$$

and, more generally,

$$\frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f\left(\frac{1}{n}\sum_{i=1}^n \delta_{x_i}\right)} \mathrm{d}\gamma_n(x_1, \dots, x_n) \text{ converges to } -\inf\{f + \text{something}\}$$

for every bounded continuous function $f : \mathcal{P}(M) \to \mathbb{R}$. The conditions for $(W_n)_n$ would be part of the "something" in the limit. The simplest version is the following. Suppose that

- $W(\mu) = \lim_{n\to\infty} \int_{M^n} W_n \mathrm{d}\mu^{\otimes n}$ exists for every probability measure $\mu \in \mathcal{P}(M)$ and that,
- whenever $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ converges to μ, the inequality $\liminf W_n(x_1, \dots, x_n) \geq W(\mu)$ holds.

Suppose, in addition, that W_n is uniformly bounded from below and that β_n converges to some $\beta \in (0, \infty)$. Then, the following holds for every bounded continuous function $f : \mathcal{P}(M) \to \mathbb{R}$.

$$\lim_{n\to\infty} \frac{1}{n\beta_n} \log \int_{M^n} e^{-n\beta_n f\circ i_n} \mathrm{d}\gamma_n = -\inf_{\mu \ll \sigma} \left\{ f(\mu) + W(\mu) + \frac{1}{\beta} \int_M (\mu \log \mu) \mathrm{d}\sigma \right\}.$$

Here, $i_n : M^n \to \mathcal{P}(M)$ is given by $i_n(x_1, \dots, x_n) = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ and the symbol μ in $\mu \log \mu$ actually denotes the density of μ with respect to σ.

There is a version where $\beta = \infty$. In this case, the entropy term $\frac{1}{\beta}\int_M (\mu \log \mu)\mathrm{d}\sigma$ disappears so that we need two more conditions. One of this conditions is to make the entropy term actually disappear and it says that,

- for every $\mu \in \mathcal{P}(M)$, there exists a sequence $(\mu_n)_n$ such that $\int_M (\mu_n \log \mu_n)\mathrm{d}\sigma < \infty$ and $\lim_{n\to\infty} W(\mu_n) = W(\mu)$.

The second is a compactness condition that appears because the entropy does not help anymore. It says that,

- if $W_n(x_1, \dots, x_n)$ is uniformly bounded from above (where n is just an increasing sequence), then $\frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ has a convergent subsequence.

Remark. *I see the measure γ_n as natural from the canonical ensemble viewpoint. This is why I take a sequence $(W_n)_n$ of energies instead of focusing on some particular case. On the other hand, I believe it is nice that those conditions are simple enough to make the proof almost straightforward. The main arguments are already found in the book* A Weak Convergence Approach to the Theory of Large Deviations *by Paul Dupuis and Richard Steven Ellis.*

Idea of the proof

The main idea, part of the philosophy of the book of Dupuis and Ellis, is to notice that

$$\begin{aligned}\frac{1}{n\beta_n}\log\int_{M^n} e^{-n\beta_n f\circ i_n}\mathrm{d}\gamma_n &= \frac{1}{n\beta_n}\log\int_{M^n} e^{-n\beta_n(f\circ i_n+W_n)}\mathrm{d}\sigma^{\otimes n}\\ &= -\inf_{\tau\ll\sigma^{\otimes n}}\left\{\int_{M^n}(f\circ i_n+W_n)\mathrm{d}\tau+\frac{1}{\beta_n}\left(\frac{1}{n}\int_{M^n}(\tau\log\tau)\mathrm{d}\sigma^{\otimes n}\right)\right\}.\end{aligned}$$

Then, we need to show that the infima converge. For instance, if we take $\tau=\mu^{\otimes n}$, we have

$$\int_{M^n}(f\circ i_n+W_n)\mathrm{d}\tau+\frac{1}{\beta_n}\left(\frac{1}{n}\int_{M^n}(\tau\log\tau)\mathrm{d}\sigma^{\otimes n}\right)\longrightarrow f(\mu)+W(\mu)+\frac{1}{\beta}\int_M(\mu\log\mu)\mathrm{d}\sigma,$$

where we have used the (weak) law of large numbers for the $f\circ i_n$ part while the definition of W is used for the W_n part.