Aspects géométriques et probabilistes des gaz de Coulomb

Dauphine | PSL
UNIVERSITÉ PARIS

David García Zelada

28 Juin 2019

Geometric and probabilistic aspects of Coulomb gases

David García-Zelada

June 28, 2019

Outline

Outline

Gibbs measure of n particles

System of n (random) particles in the Polish space M.

Gibbs measure of n particles

System of n (random) particles in the Polish space M.

Definition (Gibbs measure of n particles)

$$\mathrm{d}\mathbb{P}_n = \frac{1}{\mathcal{Z}_n} \exp(-n\beta_n H_n) \mathrm{d}\sigma^{\otimes n}.$$

Gibbs measure of n particles

System of n (random) particles in the Polish space M.

Definition (Gibbs measure of n particles)

$$\mathrm{d}\mathbb{P}_n = \frac{1}{\mathcal{Z}_n}\exp(-n\beta_n H_n)\mathrm{d}\sigma^{\otimes n}.$$

- Symmetric $H_n : M^n \to (-\infty, \infty]$, n-particle energy

Gibbs measure of n particles

System of n (random) particles in the Polish space M.

Definition (Gibbs measure of n particles)

$$\mathrm{d}\mathbb{P}_n = \frac{1}{\mathcal{Z}_n}\exp(-n\beta_n H_n)\mathrm{d}\sigma^{\otimes n}.$$

- Symmetric $H_n : M^n \to (-\infty, \infty]$, n-particle energy,
- $\sigma \in \mathcal{P}(M)$, probability measure on M

Gibbs measure of n particles

System of n (random) particles in the Polish space M.

Definition (Gibbs measure of n particles)

$$\mathrm{d}\mathbb{P}_n = \frac{1}{\mathcal{Z}_n} \exp(-n\beta_n H_n) \mathrm{d}\sigma^{\otimes n}.$$

- Symmetric $H_n : M^n \to (-\infty, \infty]$, n-particle energy,
- $\sigma \in \mathcal{P}(M)$, probability measure on M,
- $\beta_n \in [0, \infty)$, inverse temperature

Gibbs measure of n particles

System of n (random) particles in the Polish space M.

Definition (Gibbs measure of n particles)

$$\mathrm{d}\mathbb{P}_n = \frac{1}{\mathcal{Z}_n} \exp(-n\beta_n H_n)\mathrm{d}\sigma^{\otimes n}.$$

- Symmetric $H_n : M^n \to (-\infty, \infty]$, n-particle energy,
- $\sigma \in \mathcal{P}(M)$, probability measure on M,
- $\beta_n \in [0, \infty)$, inverse temperature and
- $\mathcal{Z}_n = \int_{M^n} \exp(-n\beta_n H_n)\mathrm{d}\sigma^{\otimes n} \in (0, \infty)$, partition function.

Example: Two-body interaction model

$G : M \times M \to (-\infty, \infty]$ bounded from below,

$$H_n(x_1, \ldots, x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j).$$

Question: macroscopic behavior

$$(X_1, \ldots, X_n) \sim \mathbb{P}_n \quad \text{and} \quad \hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}.$$

Question: macroscopic behavior

$$(X_1, \ldots, X_n) \sim \mathbb{P}_n \quad \text{and} \quad \hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}.$$

How $\{H_n\}_{n\in\mathbb{N}}$ determines the limit of $\{\hat{\mu}_n\}_{n\in\mathbb{N}}$?

Notion of limit

- Sequence $\{H_n\}_{n\in\mathbb{N}}$ **uniformly bounded from below**

Notion of limit

- Sequence $\{H_n\}_{n\in\mathbb{N}}$ **uniformly bounded from below** and
- $H : \mathcal{P}(M) \to (-\infty, \infty]$.

Notion of limit

- Sequence $\{H_n\}_{n\in\mathbb{N}}$ **uniformly bounded from below** and
- $H : \mathcal{P}(M) \to (-\infty, \infty]$.

Definition (Macroscopic limit)

H is the macroscopic limit of $\{H_n\}_{n\in\mathbb{N}}$ if

Notion of limit

- Sequence $\{H_n\}_{n\in\mathbb{N}}$ **uniformly bounded from below** and
- $H : \mathcal{P}(M) \to (-\infty, \infty]$.

Definition (Macroscopic limit)

H is the macroscopic limit of $\{H_n\}_{n\in\mathbb{N}}$ if

- $\forall \mu \in \mathcal{P}(M)$

$$\lim_{n\to\infty} \mathbb{E}_{\mu^{\otimes n}}[H_n] = H(\mu)$$

Notion of limit

- Sequence $\{H_n\}_{n\in\mathbb{N}}$ **uniformly bounded from below** and
- $H : \mathcal{P}(M) \to (-\infty, \infty]$.

Definition (Macroscopic limit)

H is the macroscopic limit of $\{H_n\}_{n\in\mathbb{N}}$ if

- $\forall \mu \in \mathcal{P}(M)$

$$\lim_{n\to\infty} \mathbb{E}_{\mu^{\otimes n}}[H_n] = H(\mu) \quad \text{and}$$

- whenever $\frac{1}{n}\sum_{i=1}^n \delta_{x_i} \to \mu$

$$\liminf_{n\to\infty} H_n(x_1, \dots, x_n) \geq H(\mu).$$

Example: Two-body interaction model

$G : M \times M \to (-\infty, \infty]$ bounded from below,

$$H_n(x_1, \ldots, x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j).$$

Example: Two-body interaction model

$G : M \times M \to (-\infty, \infty]$ bounded from below,

$$H_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j).$$

If G is lower semicontinuous,

$$H(\mu) = \frac{1}{2} \int_{M \times M} G(x, y) \mathrm{d}\mu(x) \mathrm{d}\mu(y)$$

is the macroscopic limit of $\{H_n\}_{n \in \mathbb{N}}$.

Relative entropy and free energy

Definition (Relative entropy)

Let $\mu \in \mathcal{P}(M)$. If $\mathrm{d}\mu = \rho\,\mathrm{d}\sigma$

$$D(\mu\|\sigma) = \int_M \rho \log \rho \,\mathrm{d}\sigma.$$

Otherwise, $D(\mu\|\sigma) = \infty$.

Relative entropy and free energy

Definition (Relative entropy)

Let $\mu \in \mathcal{P}(M)$. If $\mathrm{d}\mu = \rho\,\mathrm{d}\sigma$

$$D(\mu\|\sigma) = \int_M \rho \log \rho \,\mathrm{d}\sigma.$$

Otherwise, $D(\mu\|\sigma) = \infty$.

Definition (Free energy)

Let $\beta \in (0, \infty)$ and $H : \mathcal{P}(M) \to (-\infty, \infty]$. For $\mu \in \mathcal{P}(M)$

$$F_\beta(\mu) = H(\mu) + \frac{1}{\beta} D(\mu\|\sigma).$$

A Laplace principle

Theorem (Laplace principle for positive temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$

A Laplace principle

Theorem (Laplace principle for positive temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$ *and*
- $\{\beta_n\}_{n\in\mathbb{N}}$ *converges to some* $\beta \in (0, \infty)$.

A Laplace principle

Theorem (Laplace principle for positive temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$ *and*
- $\{\beta_n\}_{n\in\mathbb{N}}$ *converges to some* $\beta \in (0,\infty)$.

Let $I_\beta(\mu) = F_\beta(\mu) - \inf F_\beta$.

A Laplace principle

Theorem (Laplace principle for positive temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$ *and*
- $\{\beta_n\}_{n\in\mathbb{N}}$ *converges to some* $\beta \in (0,\infty)$.

Let $I_\beta(\mu) = F_\beta(\mu) - \inf F_\beta$. *Then,* $\forall$ *bounded continuous* $f : \mathcal{P}(M) \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n\beta_n} \log \mathbb{E}\left[e^{-n\beta_n f(\hat{\mu}_n)}\right] = -\inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + I_\beta(\mu)\}.$$

A Laplace principle

Theorem (Laplace principle for positive temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$ *and*
- $\{\beta_n\}_{n\in\mathbb{N}}$ *converges to some* $\beta \in (0,\infty)$.

Let $I_\beta(\mu) = F_\beta(\mu) - \inf F_\beta$. *Then,* $\forall$ *bounded continuous* $f : \mathcal{P}(M) \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n\beta_n} \log \mathbb{E}\left[e^{-n\beta_n f(\hat{\mu}_n)}\right] = -\inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + I_\beta(\mu)\}\,.$$

'Laplace principle' $\equiv$ LDP for $\hat{\mu}_n$ with rate function I_β and at speed $n\beta_n$.

A Laplace principle

Theorem (Laplace principle for positive temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$ *and*
- $\{\beta_n\}_{n\in\mathbb{N}}$ *converges to some* $\beta \in (0,\infty)$.

Let $I_\beta(\mu) = F_\beta(\mu) - \inf F_\beta$. *Then,* $\forall$ *bounded continuous* $f : \mathcal{P}(M) \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n\beta_n} \log \mathbb{E}\left[e^{-n\beta_n f(\hat{\mu}_n)}\right] = -\inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + I_\beta(\mu)\}\,.$$

'Laplace principle' $\equiv$ LDP for $\hat{\mu}_n$ with rate function I_β and at speed $n\beta_n$.

History: [Sanov '57],
[Messer & Spohn '82], [Caglioti, Lions, Marchioro & Pulvirenti '92],
[Bodineau & Guionnet '99], [Dupuis, Laschos & Ramanan '15] and [Berman '18].

Infinite β

And when $\beta_n \to \infty$? Two more conditions.

Infinite β

And when $\beta_n \to \infty$? Two more conditions.

- $\{H_n\}_{n\in\mathbb{N}}$ **confining**:

Infinite β

And when $\beta_n \to \infty$? Two more conditions.

- $\{H_n\}_{n\in\mathbb{N}}$ **confining**: If $H_n(x_1, \ldots, x_n) < A$ uniformly on n then

$$\left\{\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}\right\}_{n\in\mathbb{N}} \quad \text{is precompact in } \mathcal{P}(M).$$

Infinite β

And when $\beta_n \to \infty$? Two more conditions.

- $\{H_n\}_{n\in\mathbb{N}}$ **confining**: If $H_n(x_1, \dots, x_n) < A$ uniformly on n then

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \right\}_{n\in\mathbb{N}} \quad \text{is precompact in } \mathcal{P}(M).$$

- H **regular**:

Infinite β

And when $\beta_n \to \infty$? Two more conditions.

- $\{H_n\}_{n\in\mathbb{N}}$ **confining**: If $H_n(x_1, \ldots, x_n) < A$ uniformly on n then

$$\left\{ \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \right\}_{n\in\mathbb{N}} \quad \text{is precompact in } \mathcal{P}(M).$$

- H **regular**: If $H(\mu) < \infty$
  $\exists \ \{\mu_n\}_{n\in\mathbb{N}}$ s.t. $\mu_n \to \mu$,

$$\forall n, D(\mu_n \| \sigma) < \infty \text{ and } \lim_{n\to\infty} H(\mu_n) = H(\mu).$$

Example: Two-body interaction model

$G : M \times M \to (-\infty, \infty]$ bounded from below,

$$H_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j).$$

Example: Two-body interaction model

$G : M \times M \to (-\infty, \infty]$ bounded from below,

$$H_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j).$$

If $G(x, y) \to \infty$ whenever $x, y \to \infty$ at the same time then

$\{H_n\}_{n\in\mathbb{N}}$ is confining.

A Laplace principle

Theorem (Laplace principle for zero temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$

A Laplace principle

Theorem (Laplace principle for zero temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$,
- $\{\beta_n\}_{n\in\mathbb{N}}$ *tends to infinity*

A Laplace principle

Theorem (Laplace principle for zero temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$*,*
- $\{\beta_n\}_{n\in\mathbb{N}}$ *tends to infinity,*
- $\{H_n\}_{n\in\mathbb{N}}$ *is confining and*
- H *is regular.*

A Laplace principle

Theorem (Laplace principle for zero temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$,
- $\{\beta_n\}_{n\in\mathbb{N}}$ *tends to infinity,*
- $\{H_n\}_{n\in\mathbb{N}}$ *is confining and*
- H *is regular.*

Let $I(\mu) = H(\mu) - \inf H$.

A Laplace principle

Theorem (Laplace principle for zero temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$,
- $\{\beta_n\}_{n\in\mathbb{N}}$ *tends to infinity,*
- $\{H_n\}_{n\in\mathbb{N}}$ *is confining and*
- H *is regular.*

Let $I(\mu) = H(\mu) - \inf H$. *Then,* $\forall$ *bounded continuous* $f : \mathcal{P}(M) \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n\beta_n} \log \mathbb{E}\left[e^{-n\beta_n f(\hat{\mu}_n)}\right] = - \inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + I(\mu)\}.$$

A Laplace principle

Theorem (Laplace principle for zero temperature [GZ])

Suppose

- H *is the macroscopic limit of* $\{H_n\}_{n\in\mathbb{N}}$,
- $\{\beta_n\}_{n\in\mathbb{N}}$ *tends to infinity,*
- $\{H_n\}_{n\in\mathbb{N}}$ *is confining and*
- H *is regular.*

Let $I(\mu) = H(\mu) - \inf H$. *Then,* $\forall$ *bounded continuous* $f : \mathcal{P}(M) \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n\beta_n} \log \mathbb{E}\left[e^{-n\beta_n f(\hat{\mu}_n)}\right] = -\inf_{\mu\in\mathcal{P}(M)} \{f(\mu) + I(\mu)\}.$$

History: [Ben Arous & Guionnet '97], [Ben Arous & Zeitouni '98], [Hiai & Petz '98], [Zeitouni & Zelditch '10], [Hardy '12], [Chafaï Gozlan & Zitt '14], [Dupuis, Laschos & Ramanan '15] and [Berman '18].

Idea of the proof

Motivation: Weak convergence approach to large deviations of Dupuis and Ellis.

Idea of the proof

Let $i_n : M^n \to \mathcal{P}(M)$ be

$$i_n(x_1, \cdots, x_n) = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$$

Idea of the proof

Let $i_n : M^n \to \mathcal{P}(M)$ be

$$i_n(x_1, \cdots, x_n) = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}$$

and write

$$\frac{1}{n\beta_n} \log \int_{M^n} \exp\left(-n\beta_n \left\{f \circ i_n + H_n\right\}\right) \mathrm{d}\sigma^{\otimes n}$$
$$= - \inf_{\nu \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_\nu[f \circ i_n + H_n] + \frac{1}{n\beta_n} D(\nu \| \sigma^{\otimes n}) \right\}$$

Idea of the proof

Let $i_n : M^n \to \mathcal{P}(M)$ be

$$i_n(x_1, \cdots, x_n) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$$

and write

$$\frac{1}{n\beta_n} \log \int_{M^n} \exp\left(-n\beta_n \left\{f \circ i_n + H_n\right\}\right) \mathrm{d}\sigma^{\otimes n}$$
$$= - \inf_{\nu \in \mathcal{P}(M^n)} \left\{ \mathbb{E}_\nu[f \circ i_n + H_n] + \frac{1}{n\beta_n} D(\nu \| \sigma^{\otimes n}) \right\}$$

which converges to

$$- \inf_{\mu \in \mathcal{P}(M)} \left\{ f(\mu) + H(\mu) + \frac{1}{\beta} D(\mu \| \sigma) \right\}.$$

Idea of the proof

Let $i_n : M^n \to \mathcal{P}(M)$ be

$$i_n(x_1, \cdots, x_n) = \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}$$

and write

$$\begin{aligned}
&\frac{1}{n\beta_n}\log\int_{M^n} \exp\left(-n\beta_n\left\{f \circ i_n + H_n\right\}\right) \mathrm{d}\sigma^{\otimes n} \\
&= -\inf_{\nu\in\mathcal{P}(M^n)}\left\{\mathbb{E}_\nu[f\circ i_n + H_n] + \frac{1}{n\beta_n}D(\nu\|\sigma^{\otimes n})\right\}
\end{aligned}$$

which converges to

$$-\inf_{\mu\in\mathcal{P}(M)}\left\{f(\mu) + H(\mu) + \frac{1}{\beta}D(\mu\|\sigma)\right\}.$$

Use macroscopic convergence and properties of the entropy!

Outline

Coulomb gas on a compact manifold

- $M =$ compact oriented Riemannian manifold

Coulomb gas on a compact manifold

- M = compact oriented Riemannian manifold,
- σ = volume measure on M (of unit mass)

Coulomb gas on a compact manifold

- M = compact oriented Riemannian manifold,
- σ = volume measure on M (of unit mass) and
- G = Green function associated to some signed measure Λ.

Coulomb gas on a compact manifold

- M = compact oriented Riemannian manifold,
- σ = volume measure on M (of unit mass) and
- G = Green function associated to some signed measure Λ.

Definition (Green function for a compact manifold)

Let Λ be a differentiable signed measure on M s.t. $\Lambda(M) = 1$.
$G : M \times M \to (-\infty, \infty]$ continuous s.t.

$$\Delta G(x, \cdot) = -\delta_x + \Lambda.$$

Coulomb gas on a compact manifold

- M = compact oriented Riemannian manifold,
- σ = volume measure on M (of unit mass) and
- G = Green function associated to some signed measure Λ.

Definition (Green function for a compact manifold)

Let Λ be a differentiable signed measure on M s.t. $\Lambda(M) = 1$.
$G : M \times M \to (-\infty, \infty]$ continuous s.t.

$$\Delta G(x, \cdot) = -\delta_x + \Lambda.$$

- $H_n(x_1, \ldots, x_n) = \frac{1}{n^2} \sum_{i<j} G(x_i, x_j)$.

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log|z-w| \mathrm{d}\nu(w).$$

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log |z - w| \mathrm{d}\nu(w).$$

Denote by η the standard metric on $\mathbb{C}$.

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log|z - w| \mathrm{d}\nu(w).$$

Denote by η the standard metric on $\mathbb{C}$.

- $M = S^2 = \mathbb{C} \cup \{\infty\}$ endowed with $g = e^{-4V}\eta$

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log|z-w| \mathrm{d}\nu(w).$$

Denote by η the standard metric on $\mathbb{C}$.

- $M = S^2 = \mathbb{C} \cup \{\infty\}$ endowed with $g = e^{-4V}\eta$,
- $\mathrm{d}\sigma = e^{-4V}\mathrm{d}\ell_{\mathbb{C}}$ normalized

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log|z-w| \mathrm{d}\nu(w).$$

Denote by η the standard metric on $\mathbb{C}$.

- $M = S^2 = \mathbb{C} \cup \{\infty\}$ endowed with $g = e^{-4V}\eta$,
- $\mathrm{d}\sigma = e^{-4V}\mathrm{d}\ell_{\mathbb{C}}$ normalized,
- $\Lambda \;= \nu$

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log|z - w| \mathrm{d}\nu(w).$$

Denote by η the standard metric on $\mathbb{C}$.

- $M = S^2 = \mathbb{C} \cup \{\infty\}$ endowed with $g = e^{-4V}\eta$,
- $\mathrm{d}\sigma = e^{-4V}\mathrm{d}\ell_{\mathbb{C}}$ normalized,
- $\Lambda \ = \nu = \frac{1}{2\pi}\Delta V$

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log|z-w| \mathrm{d}\nu(w).$$

Denote by η the standard metric on $\mathbb{C}$.

- $M = S^2 = \mathbb{C} \cup \{\infty\}$ endowed with $g = e^{-4V}\eta$,
- $\mathrm{d}\sigma = e^{-4V}\mathrm{d}\ell_{\mathbb{C}}$ normalized,
- $\Lambda \ = \nu = \frac{1}{2\pi}\Delta V = \frac{1}{4\pi}\mathrm{Ric}_g$

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log|z-w| \mathrm{d}\nu(w).$$

Denote by η the standard metric on $\mathbb{C}$.

- $M = S^2 = \mathbb{C} \cup \{\infty\}$ endowed with $g = e^{-4V}\eta$,
- $\mathrm{d}\sigma = e^{-4V}\mathrm{d}\ell_{\mathbb{C}}$ normalized,
- $\Lambda = \nu = \frac{1}{2\pi}\Delta V = \frac{1}{4\pi}\mathrm{Ric}_g$ and
- $G(z,w) = -\frac{1}{2\pi}\log|z-w| + \frac{1}{2\pi}V(z) + \frac{1}{2\pi}V(w)$

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log|z-w| \mathrm{d}\nu(w).$$

Denote by η the standard metric on $\mathbb{C}$.

- $M = S^2 = \mathbb{C} \cup \{\infty\}$ endowed with $g = e^{-4V}\eta$,
- $\mathrm{d}\sigma = e^{-4V}\mathrm{d}\ell_{\mathbb{C}}$ normalized,
- $\Lambda = \nu = \frac{1}{2\pi}\Delta V = \frac{1}{4\pi}\mathrm{Ric}_g$ and
- $G(z,w) = -\frac{1}{2\pi}\log|z-w| + \frac{1}{2\pi}V(z) + \frac{1}{2\pi}V(w)$

so that

$$\Delta G(z,\cdot) = -\delta_z + \nu.$$

Example: Standard 2D Coulomb gas

Let $\nu \in \mathcal{P}(\mathbb{C})$ and

$$V(z) = \int_{\mathbb{C}} \log |z - w| \mathrm{d}\nu(w).$$

Denote by η the standard metric on $\mathbb{C}$.

- $M = S^2 = \mathbb{C} \cup \{\infty\}$ endowed with $g = e^{-4V}\eta$,
- $\mathrm{d}\sigma = e^{-4V}\mathrm{d}\ell_{\mathbb{C}}$ normalized,
- $\Lambda = \nu = \frac{1}{2\pi}\Delta V = \frac{1}{4\pi}\mathrm{Ric}_g$ and
- $G(z,w) = -\frac{1}{2\pi}\log|z-w| + \frac{1}{2\pi}V(z) + \frac{1}{2\pi}V(w)$

so that

$$\Delta G(z,\cdot) = -\delta_z + \nu.$$

Then

$$\mathrm{d}\mathbb{P}_n(z_1,\ldots,z_n) = \frac{1}{\mathcal{Z}_n}\prod_{i<j}|z_i - z_j|^{\frac{\beta n}{2\pi n}} e^{-\left(\frac{\beta n}{2\pi n}(n-1)+4\right)\sum_{i=1}^n V(z_i)} \mathrm{d}\ell_{\mathbb{C}^n}(z_1,\ldots,z_n).$$

Convergence of empirical measures

If $\beta_n \to \infty$ and $(X_1, \dots, X_n) \sim \mathbb{P}_n$ then

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} \xrightarrow{a.s.} \mu_{\text{eq}} \quad \text{where} \quad \mu_{\text{eq}} = \arg\min H.$$

Convergence of empirical measures

If $\beta_n \to \infty$ and $(X_1, \ldots, X_n) \sim \mathbb{P}_n$ then

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} \xrightarrow{a.s.} \mu_{\text{eq}} \quad \text{where} \quad \mu_{\text{eq}} = \arg\min H.$$

Definition (Kantorovich-Wasserstein distance)

$\mathrm{W}_1 : \mathcal{P}(M) \times \mathcal{P}(M) \to [0, \infty)$

$$\mathrm{W}_1(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}\left[d(X, Y)\right].$$

Convergence of empirical measures

If $\beta_n \to \infty$ and $(X_1, \ldots, X_n) \sim \mathbb{P}_n$ then

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} \xrightarrow{a.s.} \mu_{\mathrm{eq}} \quad \text{where} \quad \mu_{\mathrm{eq}} = \arg\min H.$$

Definition (Kantorovich-Wasserstein distance)

$\mathrm{W}_1 : \mathcal{P}(M) \times \mathcal{P}(M) \to [0, \infty)$

$$\mathrm{W}_1(\mu, \nu) = \inf_{X \sim \mu, Y \sim \nu} \mathbb{E}\left[d(X, Y)\right].$$

This distance metrizes the weak convergence!

$$\mathrm{W}_1(\hat{\mu}_n, \mu_{\mathrm{eq}}) \xrightarrow{a.s.} 0.$$

Question: concentration of measure

LDP implies $\exists \mathcal{I}_r > 0$ and $\exists \varepsilon_n$ s.t. $\varepsilon_n \to 0$

$$\mathbb{P}\left(\mathrm{W}_1(\hat{\mu}_n, \mu_{\mathrm{eq}}) \geq r\right) \leq \exp\left(-n\beta_n \mathcal{I}_r + n\beta_n \varepsilon_n\right).$$

Question: concentration of measure

LDP implies $\exists \mathcal{I}_r > 0$ and $\exists \varepsilon_n$ s.t. $\varepsilon_n \to 0$

$$\mathbb{P}\left(\mathrm{W}_1(\hat{\mu}_n, \mu_{\mathrm{eq}}) \geq r\right) \leq \exp\left(-n\beta_n \mathcal{I}_r + n\beta_n \varepsilon_n\right).$$

Can we find a simple $\mathcal{I}_r$ and ε_n?

Concentration for Coulomb gases

Theorem (Concentration inequality [GZ])

If $\dim M = 2$ *then* $\exists C = C(M, \Lambda)$ *s.t.*

$$\mathbb{P}(\mathrm{W}_1(\hat{\mu}_n, \mu_{\mathrm{eq}}) \geq r) \leq \exp\left(-n\beta_n \frac{r^2}{4} + \frac{\beta_n}{8\pi}\log(n) + nD(\mu_{\mathrm{eq}} \| \sigma) + C\beta_n\right).$$

Concentration for Coulomb gases

Theorem (Concentration inequality [GZ])

If $\dim M = 2$ *then* $\exists C = C(M, \Lambda)$ *s.t.*

$$\mathbb{P}(\mathrm{W}_1(\hat{\mu}_n, \mu_{\mathrm{eq}}) \geq r) \leq \exp\left(-n\beta_n \frac{r^2}{4} + \frac{\beta_n}{8\pi}\log(n) + nD(\mu_{\mathrm{eq}}\|\sigma) + C\beta_n\right).$$

If $\dim M \geq 3$ *then* $\exists C = C(M, \Lambda)$ *s.t.*

$$\mathbb{P}(\mathrm{W}_1(\hat{\mu}_n, \mu_{\mathrm{eq}}) \geq r) \leq \exp\left(-n\beta_n \frac{r^2}{4} + nD(\mu_{\mathrm{eq}}\|\sigma) + C\beta_n n^{1-2/\dim M}\right).$$

Concentration for Coulomb gases

Theorem (Concentration inequality [GZ])

If $\dim M = 2$ *then* $\exists C = C(M, \Lambda)$ *s.t.*

$$\mathbb{P}(\mathrm{W}_1(\hat{\mu}_n, \mu_{\mathrm{eq}}) \geq r) \leq \exp\left(-n\beta_n \frac{r^2}{4} + \frac{\beta_n}{8\pi}\log(n) + nD(\mu_{\mathrm{eq}} \| \sigma) + C\beta_n\right).$$

If $\dim M \geq 3$ *then* $\exists C = C(M, \Lambda)$ *s.t.*

$$\mathbb{P}(\mathrm{W}_1(\hat{\mu}_n, \mu_{\mathrm{eq}}) \geq r) \leq \exp\left(-n\beta_n \frac{r^2}{4} + nD(\mu_{\mathrm{eq}} \| \sigma) + C\beta_n n^{1-2/\dim M}\right).$$

History: [Maïda & Maurel-Segala '12], [Rougerie & Serfaty '16], [Chafaï, Hardy & Maïda '18] and [Berman '19].

Idea of the proof

Idea of the proof

Lemma (Energy-distance inequality)

$$\frac{1}{2}\mathrm{W}_1(\mu, \mu_{\mathrm{eq}})^2 \leq H(\mu) - H(\mu_{\mathrm{eq}}).$$

Idea of the proof

Lemma (Energy-distance inequality)

$$\frac{1}{2}\mathrm{W}_1(\mu, \mu_{\mathrm{eq}})^2 \leq H(\mu) - H(\mu_{\mathrm{eq}}).$$

Lemma (Regularization in two dimensions)

$\exists R_n : M^n \to \mathcal{P}(M)$ and $C > 0$ s.t.

$$H_n(x_1, \dots, x_n) \geq H(R_n(x_1, \dots, x_n)) - \frac{1}{8\pi n}\log(n) - \frac{C}{n} \quad \textit{and}$$

$$\mathrm{W}_1\left(R_n(x_1, \dots, x_n), \frac{1}{n}\sum_{i=1}^{n} \delta_{x_i}\right) \leq \frac{C}{\sqrt{n}}.$$

Idea of the proof

Lemma (Energy-distance inequality)

$$\frac{1}{2}\mathrm{W}_1(\mu,\mu_{\mathrm{eq}})^2 \leq H(\mu) - H(\mu_{\mathrm{eq}}).$$

Lemma (Regularization)

$\exists R_n : M^n \to \mathcal{P}(M)$ *and* $C > 0$ *s.t.*

$$H_n(x_1,\ldots,x_n) \geq H(R_n(x_1,\ldots,x_n)) - \frac{1}{8\pi n}\log(n)1_{\dim M=2} - \frac{C}{n^{2/\dim M}} \quad \textit{and}$$

$$\mathrm{W}_1\left(R_n(x_1,\ldots,x_n), \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}\right) \leq \frac{C}{n^{1/\dim M}}.$$

Choice of R_n

Let $p_t(x,y)$ be the heat kernel and

$$\mathrm{d}\mu_x^t = p_t(x,y)\mathrm{d}\sigma(y).$$

Choice of R_n

Let $p_t(x, y)$ be the heat kernel and

$$\mathrm{d}\mu_x^t = p_t(x, y)\mathrm{d}\sigma(y).$$

Define

$$R^t(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} \mu_{x_i}^t$$

Choice of R_n

Let $p_t(x,y)$ be the heat kernel and

$$\mathrm{d}\mu_x^t = p_t(x,y)\mathrm{d}\sigma(y).$$

Define

$$R^t(x_1,\dots,x_n) = \frac{1}{n}\sum_{i=1}^{n}\mu_{x_i}^t$$

and let

$$\boxed{R_n = R^t \quad \text{for} \quad t = n^{-2/\dim M}.}$$

Outline

Determinantal case

Let $\nu \in \mathcal{P}(\mathbb{C})$ be rotationally invariant so that

$$V(z) = \int_1^{|z|} \frac{\nu(D_s)}{s} \mathrm{d}s.$$

Determinantal case

Let $\nu \in \mathcal{P}(\mathbb{C})$ be rotationally invariant so that

$$V(z) = \int_1^{|z|} \frac{\nu(D_s)}{s} \mathrm{d}s.$$

If $\beta_n = 4\pi n$ then

$$\mathrm{d}\mathbb{P}_n(z_1, \ldots, z_n) = \frac{1}{\mathcal{Z}_n} \prod_{i<j} |z_i - z_j|^2 e^{-2(n+1)\sum_{i=1}^n V(z_i)} \mathrm{d}\ell_{\mathbb{C}^n}(z_1, \ldots, z_n)$$

and the point process is determinantal!

Determinantal case

Let $\nu \in \mathcal{P}(\mathbb{C})$ be rotationally invariant so that

$$V(z) = \int_1^{|z|} \frac{\nu(D_s)}{s} \mathrm{d}s.$$

If $\beta_n = 4\pi n$ then

$$\mathrm{d}\mathbb{P}_n(z_1, \ldots, z_n) = \frac{1}{\mathcal{Z}_n} \prod_{i<j} |z_i - z_j|^2 e^{-2(n+1)\sum_{i=1}^n V(z_i)} \mathrm{d}\ell_{\mathbb{C}^n}(z_1, \ldots, z_n)$$

and the point process is determinantal!
If $(X_1, \ldots, X_n) \sim \mathbb{P}_n$ and $f : \mathbb{C} \to \mathbb{R}$ is bounded continuous then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = \int_{\mathbb{C}} f \mathrm{d}\nu.$$

Question: uncharged regions

If $\operatorname{supp} f \cap \operatorname{supp} \nu = \emptyset$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = 0.$$

Question: uncharged regions

If $\operatorname{supp} f \cap \operatorname{supp} \nu = \emptyset$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = 0.$$

What is $\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i)$?

Possible cases

Connected components of $\mathbb{C} \setminus \operatorname{supp} \nu$:

Possible cases

Connected components of $\mathbb{C} \setminus \operatorname{supp} \nu$:
A family of possibilities (up to rescaling)

Possible cases

Connected components of $\mathbb{C} \setminus \operatorname{supp} \nu$:
A family of possibilities (up to rescaling)

- $\mathbb{D} = D_1$

Possible cases

Connected components of $\mathbb{C} \setminus \operatorname{supp} \nu$:
A family of possibilities (up to rescaling)

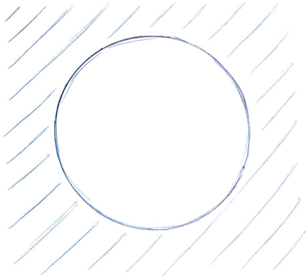
- $\mathbb{D} = D_1$,
- $\mathbb{C} \setminus \bar{\mathbb{D}}$

Possible cases

Connected components of $\mathbb{C} \setminus \operatorname{supp} \nu$:
A family of possibilities (up to rescaling)

- $\mathbb{D} = D_1$,
- $\mathbb{C} \setminus \bar{\mathbb{D}}$ and
- $\mathbb{A}_R = D_R \setminus \bar{\mathbb{D}}$ for $R > 1$.

$\mathbb{D}$ is a connected component



$\mathbb{D}$ is a connected component

Theorem (Limiting Bergman point process [Butez & GZ])

$\forall\ f : \mathbb{C} \to \mathbb{R}$ *continuous and s.t.* $\operatorname{supp} f \subset \mathbb{D}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i) = \sum_{z \in \mathcal{B}_{\mathbb{D}}} f(z)$$

in law

$\mathbb{D}$ is a connected component

Theorem (Limiting Bergman point process [Butez & GZ])

$\forall\ f : \mathbb{C} \to \mathbb{R}$ *continuous and s.t.* $\operatorname{supp} f \subset \mathbb{D}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i) = \sum_{z \in \mathcal{B}_{\mathbb{D}}} f(z)$$

in law where $\mathcal{B}_{\mathbb{D}}$ *is a DPP on* $\mathbb{D}$ *with kernel*

$$K(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}$$

with respect to Lebesgue measure.

$\mathbb{D}$ is a connected component

Theorem (Limiting Bergman point process [Butez & GZ])

$\forall\ f : \mathbb{C} \to \mathbb{R}$ *continuous and s.t.* $\operatorname{supp} f \subset \mathbb{D}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i) = \sum_{z\in\mathcal{B}_{\mathbb{D}}} f(z)$$

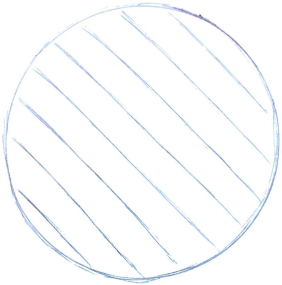
in law where $\mathcal{B}_{\mathbb{D}}$ *is a DPP on* $\mathbb{D}$ *with kernel*

$$K(z,w) = \frac{1}{\pi(1-z\bar{w})^2}$$

with respect to Lebesgue measure.

History: [Ameur, Kang & Makarov '18].

$\mathbb{C} \setminus \bar{\mathbb{D}}$ is a connected component



$\mathbb{C} \setminus \bar{\mathbb{D}}$ is a connected component

Theorem (Limiting inverse Bergman point process [Butez & GZ])

$\forall\ f : \mathbb{C} \to \mathbb{R}$ *continuous with a finite limit at* ∞ *and s.t.* $\operatorname{supp} f \subset \mathbb{C} \setminus \bar{\mathbb{D}}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i) = \sum_{z\in\mathcal{B}_{\mathbb{D}}} f(1/z)$$

in law

$\mathbb{C} \setminus \bar{\mathbb{D}}$ is a connected component

Theorem (Limiting inverse Bergman point process [Butez & GZ])

$\forall\ f : \mathbb{C} \to \mathbb{R}$ *continuous with a finite limit at* ∞ *and s.t.* $\operatorname{supp} f \subset \mathbb{C} \setminus \bar{\mathbb{D}}$

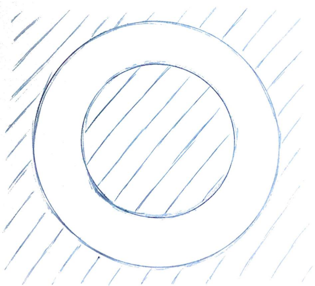
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i) = \sum_{z \in \mathcal{B}_{\mathbb{D}}} f(1/z)$$

in law where $\mathcal{B}_{\mathbb{D}}$ *is a DPP on* $\mathbb{D}$ *with kernel*

$$K(z,w) = \frac{1}{\pi(1 - z\bar{w})^2}$$

with respect to Lebesgue measure.

$\mathbb{A}_R$ is a connected component



$\mathbb{A}_R$ is a connected component

Theorem (Limiting Bergman point processes on $\mathbb{A}_R$ [GZ])

Let $q = \mu(\bar{\mathbb{D}})$ and take a subsequence of integers s.t.

$$\exp\left(2\pi(n+1)q\, i\right) \to \exp\left(2\pi\gamma\, i\right)$$

for some $\gamma \in \mathbb{R}$.

$\mathbb{A}_R$ is a connected component

Theorem (Limiting Bergman point processes on $\mathbb{A}_R$ [GZ])

Let $q = \mu(\bar{\mathbb{D}})$ *and take a subsequence of integers s.t.*

$$\exp\left(2\pi(n+1)q\, i\right) \to \exp\left(2\pi\gamma\, i\right)$$

for some $\gamma \in \mathbb{R}$*. Then* $\forall\ f : \mathbb{C} \to \mathbb{R}$ *continuous and s.t.* $\operatorname{supp} f \subset \mathbb{A}_R$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i) = \sum_{z \in \mathcal{B}_{\mathbb{A}_R,\gamma}} f(z)$$

in law

$\mathbb{A}_R$ is a connected component

Theorem (Limiting Bergman point processes on $\mathbb{A}_R$ [GZ])

Let $q = \mu(\bar{\mathbb{D}})$ *and take a subsequence of integers s.t.*

$$\exp\left(2\pi(n+1)q\, i\right) \to \exp\left(2\pi\gamma\, i\right)$$

for some $\gamma \in \mathbb{R}$*. Then* $\forall\ f : \mathbb{C} \to \mathbb{R}$ *continuous and s.t.* $\operatorname{supp} f \subset \mathbb{A}_R$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i) = \sum_{z \in \mathcal{B}_{\mathbb{A}_R,\gamma}} f(z)$$

in law where $\mathcal{B}_{\mathbb{A}_R,\gamma}$ *is a DPP on* $\mathbb{A}_R$ *associated to the kernel of*

the orthogonal projection onto the space of holomorphic functions of $\mathbb{A}_R$

with weight $|z|^{-2\gamma}$*.*

Idea of the proof

$\sum_{i=1}^n \delta_{X_i}$ is a DPP with kernel

$$K_n(z,w) = \sum_{l=0}^{n-1} a_l z^l \bar{w}^l e^{-(n+1)V(z)} e^{-(n+1)V(w)}$$

where

$$a_l = \left(\int_{\mathbb{C}} |z|^{2l} e^{-2(n+1)V(z)} \mathrm{d}\ell_{\mathbb{C}}(z) \right)^{-1}.$$

Idea of the proof

$\sum_{i=1}^{n} \delta_{X_i}$ is a DPP with kernel

$$K_n(z,w) = \sum_{l=0}^{n-1} a_l z^l \bar{w}^l e^{-(n+1)V(z)} e^{-(n+1)V(w)}$$

where

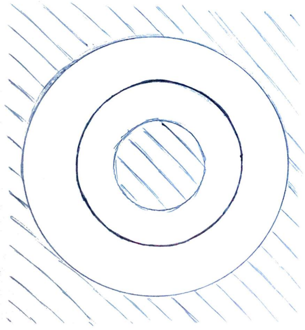
$$a_l = \left(\int_{\mathbb{C}} |z|^{2l} e^{-2(n+1)V(z)} \mathrm{d}\ell_{\mathbb{C}}(z) \right)^{-1}.$$

For the annulus,

$$\sum_{k=-\lfloor nq \rfloor}^{n-1-\lfloor nq \rfloor} a_{\lfloor nq \rfloor + k}\, z^k \bar{w}^k |z|^{\lfloor nq \rfloor} |w|^{\lfloor nq \rfloor} e^{-(n+1)V(z)} e^{-(n+1)V(w)}.$$

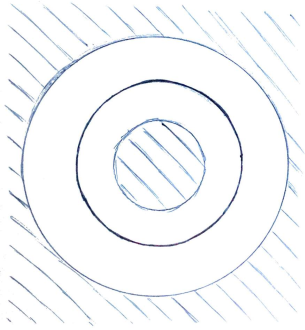
Behavior between two uncharged regions

Suppose there are $R_1 < 1 < R_2$ s.t. $\nu\left(R_1 < |z| < R_2\right) = \nu(\partial\mathbb{D}) > 0$



Behavior between two uncharged regions

Suppose there are $R_1 < 1 < R_2$ s.t. $\nu\left(R_1 < |z| < R_2\right) = \nu(\partial\mathbb{D}) > 0$ and let $q = \nu(\mathbb{D})$ and $Q = \nu(\bar{\mathbb{D}})$.



Behavior between two uncharged regions

Suppose there are $R_1 < 1 < R_2$ s.t. $\nu\left(R_1 < |z| < R_2\right) = \nu(\partial\mathbb{D}) > 0$ and let $q = \nu(\mathbb{D})$ and $Q = \nu(\bar{\mathbb{D}})$.

Theorem (Behavior at the unit circle [GZ])

$$\lim_{n\to\infty} \sum_{i=1}^{n} \delta_{n(X_i - 1)} = \xi_{q,Q}$$

where $\xi_{q,Q}$ is a DPP with kernel

$$K(z,w) = \frac{f(z)f(w)}{\pi} \int_q^Q \frac{(Q-t)(t-q)}{Q-q} e^{(z+\bar{w})t} \mathrm{d}t,$$

$$f(x+iy) = e^{-qx} \textit{ if } x < 0 \quad \textit{and} \quad f(x+iy) = e^{-Qx} \textit{ if } x \geq 0.$$

Behavior between two uncharged regions

Suppose there are $R_1 < 1 < R_2$ s.t. $\nu\left(R_1 < |z| < R_2\right) = \nu(\partial\mathbb{D}) > 0$ and let $q = \nu(\mathbb{D})$ and $Q = \nu(\bar{\mathbb{D}})$.

Theorem (Behavior at the unit circle [GZ])

$$\lim_{n\to\infty} \sum_{i=1}^{n} \delta_{n(X_i-1)} = \xi_{q,Q}$$

where $\xi_{q,Q}$ is a DPP with kernel

$$K(z,w) = \frac{f(z)f(w)}{\pi} \int_q^Q \frac{(Q-t)(t-q)}{Q-q} e^{(z+\bar{w})t} \mathrm{d}t,$$

$$f(x+iy) = e^{-qx} \text{ if } x < 0 \quad \text{and} \quad f(x+iy) = e^{-Qx} \text{ if } x \geq 0.$$

History: [Sinclair & Yattselev '12] and [Hedenmalm & Wennman '17].

Idea of the proof

Study

$$\rho_n(r) = \frac{1}{n^2}\sum_{k=0}^{n-1} a_k \left(1 + \frac{r}{n}\right)^{\left(\frac{k}{n}\right)n}$$

where

$$a_k = \left(\int_{\mathbb{C}} |z|^{2k} e^{-2(n+1)V(z)} \mathrm{d}\ell_{\mathbb{C}}(z)\right)^{-1}.$$

Idea of the proof

Study

$$\rho_n(r) = \frac{1}{n^2} \sum_{k=0}^{n-1} a_k \left(1 + \frac{r}{n}\right)^{\left(\frac{k}{n}\right)n}$$

where

$$a_k = \left(\int_{\mathbb{C}} |z|^{2k} e^{-2(n+1)V(z)} \mathrm{d}\ell_{\mathbb{C}}(z) \right)^{-1}.$$

Define $t^*(t) = \lfloor tn \rfloor / n$.

Idea of the proof

Study

$$\rho_n(r) = \frac{1}{n^2}\sum_{k=0}^{n-1} a_k \left(1+\frac{r}{n}\right)^{\left(\frac{k}{n}\right)n}$$

where

$$a_k = \left(\int_{\mathbb{C}} |z|^{2k} e^{-2(n+1)V(z)} \mathrm{d}\ell_{\mathbb{C}}(z)\right)^{-1}.$$

Define $t^*(t) = \lfloor tn \rfloor / n$. Then

$$\rho_n(r) = \int_0^1 \frac{a_{t^* n}}{n}\left(1+\frac{r}{n}\right)^{t^* n} \mathrm{d}t$$

and take the limit!

Behavior of the maxima

Theorem (Compactly supported background [Butez & GZ])

If $\nu(\mathbb{C} \setminus D_R) = 0$ *and* $\partial D_R \subset \operatorname{supp} \nu$,

$$\forall r \geq R, \quad \lim_{n \to \infty} \mathbb{P}\left(\max\{|X_1|, \dots, |X_n|\} < r\right) = \prod_{k=1}^{\infty} (1 - R^{2k} r^{-2k}).$$

Behavior of the maxima

Theorem (Compactly supported background [Butez & GZ])

If $\nu(\mathbb{C} \setminus D_R) = 0$ *and* $\partial D_R \subset \operatorname{supp} \nu$,

$$\forall r \geq R, \quad \lim_{n\to\infty} \mathbb{P}\left(\max\{|X_1|, \ldots, |X_n|\} < r\right) = \prod_{k=1}^{\infty} (1 - R^{2k} r^{-2k}).$$

Theorem (Non-compactly supported background [Butez & GZ])

If $\lim_{r\to\infty} r^{\alpha} \nu(\mathbb{C} \setminus D_r) = \lambda$,

$$\forall r > 0, \quad \lim_{n\to\infty} \mathbb{P}\left(n^{1/\alpha} \max\{|X_1|, \ldots, |X_n|\} < r\right) = \prod_{k=1}^{\infty} \frac{\Gamma\left(\frac{2k}{\alpha}, \frac{2\lambda}{\alpha} r^{-\alpha}\right)}{\Gamma\left(\frac{2k}{\alpha}\right)}.$$

Behavior of the maxima

Theorem (Compactly supported background [Butez & GZ])

If $\nu(\mathbb{C} \setminus D_R) = 0$ *and* $\partial D_R \subset \operatorname{supp} \nu$,

$$\forall r \geq R, \quad \lim_{n\to\infty} \mathbb{P}\left(\max\{|X_1|, \ldots, |X_n|\} < r\right) = \prod_{k=1}^{\infty} (1 - R^{2k} r^{-2k}).$$

Theorem (Non-compactly supported background [Butez & GZ])

If $\lim_{r\to\infty} r^{\alpha} \nu(\mathbb{C} \setminus D_r) = \lambda$,

$$\forall r > 0, \quad \lim_{n\to\infty} \mathbb{P}\left(n^{1/\alpha} \max\{|X_1|, \ldots, |X_n|\} < r\right) = \prod_{k=1}^{\infty} \frac{\Gamma\left(\frac{2k}{\alpha}, \frac{2\lambda}{\alpha} r^{-\alpha}\right)}{\Gamma\left(\frac{2k}{\alpha}\right)}.$$

History: [Rider '03], [Chafaï & Peche '14] and [Jiang & Qi '17].

Outline

Macroscopic behavior

Recall the Coulomb gas law

$$\mathrm{d}\mathbb{P}_n(z_1,\ldots,z_n) = \frac{1}{\mathcal{Z}_n}\prod_{i<j}|z_i - z_j|^{\frac{\beta n}{2\pi n}} e^{-\left(\frac{\beta n}{2\pi n}(n-1)+4\right)\sum_{i=1}^n V(z_i)}\mathrm{d}\ell_{\mathbb{C}^n}(z_1,\ldots,z_n).$$

Macroscopic behavior

Recall the Coulomb gas law

$$\mathrm{d}\mathbb{P}_n(z_1,\ldots,z_n) = \frac{1}{\mathcal{Z}_n}\prod_{i<j}|z_i - z_j|^{\frac{\beta_n}{2\pi n}} e^{-\left(\frac{\beta_n}{2\pi n}(n-1)+4\right)\sum_{i=1}^n V(z_i)}\mathrm{d}\ell_{\mathbb{C}^n}(z_1,\ldots,z_n).$$

It makes sense for any $\beta_n > -8\pi$!

Macroscopic behavior

Recall the Coulomb gas law

$$\mathrm{d}\mathbb{P}_n(z_1,\ldots,z_n)=\frac{1}{\mathcal{Z}_n}\prod_{i<j}|z_i-z_j|^{\frac{\beta_n}{2\pi n}}e^{-\left(\frac{\beta_n}{2\pi n}(n-1)+4\right)\sum_{i=1}^n V(z_i)}\mathrm{d}\ell_{\mathbb{C}^n}(z_1,\ldots,z_n).$$

It makes sense for any $\beta_n > -8\pi$!

What happens when $\beta_n \downarrow -8\pi$?

Fluctuations

Suppose the background measure on M is $\Lambda = \sigma$ so that if

$$(X_1, \ldots, X_n) \sim \text{ Coulomb gas on } M$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} = \sigma.$$

Fluctuations

Suppose the background measure on M is $\Lambda = \sigma$ so that if

$$(X_1, \ldots, X_n) \sim \text{ Coulomb gas on } M$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} = \sigma.$$

Suppose $\int_M f \mathrm{d}\sigma = 0$ and $\beta_n = n\beta$.

$$\text{What is } \lim_{n\to\infty} \sum_{i=1}^{n} f(X_i)?.$$

Uncharged region

Suppose $\operatorname{supp} f \cap \operatorname{supp} \nu = \emptyset$. We know

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i)$$

in the determinantal case.

Uncharged region

Suppose $\operatorname{supp} f \cap \operatorname{supp} \nu = \emptyset$. We know

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i)$$

in the determinantal case.
Three open questions:

Uncharged region

Suppose $\operatorname{supp} f \cap \operatorname{supp} \nu = \emptyset$. We know

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i)$$

in the determinantal case.

Three open questions:

- What happens if ν is not radial?

Uncharged region

Suppose $\operatorname{supp} f \cap \operatorname{supp} \nu = \emptyset$. We know

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i)$$

in the determinantal case.

Three open questions:

- What happens if ν is not radial?
- What happens in the non-determinantal case?

Uncharged region

Suppose $\operatorname{supp} f \cap \operatorname{supp} \nu = \emptyset$. We know

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(X_i)$$

in the determinantal case.

Three open questions:

- What happens if ν is not radial?
- What happens in the non-determinantal case?
- What happens if we only assume $f|_{\operatorname{supp} \nu} = 0$?

Thank you for your attention!

Merci pour votre attention!

¡Gracias por su atención!

Extra frames

Proof of the annulus case

Proof Annulus case, p.1

$\sum_{i=1}^{n} \delta_{X_i}$ is a DPP with kernel

$$K_n(z,w) = \sum_{l=0}^{n-1} a_l z^l \bar{w}^l e^{-(n+1)V(z)} e^{-(n+1)V(w)}$$

where

$$a_l = \left(\int_{\mathbb{C}} |z|^{2l} e^{-2(n+1)V(z)} \mathrm{d}\ell_{\mathbb{C}}(z) \right)^{-1}.$$

Proof Annulus case, p.2

Since $q = \mu(\mathbb{D})$, we have

$$V(z) = q \log |z| \quad \text{for} \quad z \in \bar{\mathbb{A}}_R$$

and

$$V(z) > q \log |z| \quad \text{if} \quad z \notin \bar{\mathbb{A}}_R.$$

Proof Annulus case, p.2

Since $q = \mu(\mathbb{D})$, we have

$$V(z) = q \log |z| \quad \text{for} \quad z \in \bar{\mathbb{A}}_R$$

and

$$V(z) > q \log |z| \quad \text{if} \quad z \notin \bar{\mathbb{A}}_R.$$

We assume $\lim_{n\to\infty} \{(n+1)q - \lfloor nq \rfloor\} = \gamma$

Proof Annulus case, p.2

Since $q = \mu(\mathbb{D})$, we have

$$V(z) = q \log |z| \quad \text{for} \quad z \in \bar{\mathbb{A}}_R$$

and

$$V(z) > q \log |z| \quad \text{if} \quad z \notin \bar{\mathbb{A}}_R.$$

We assume $\lim_{n\to\infty} \{(n+1)q - \lfloor nq \rfloor\} = \gamma$ so that

$$\lim_{n\to\infty} |z|^{\lfloor nq \rfloor} e^{-(n+1)V(z)} = |z|^{-\gamma}.$$

Proof Annulus case, p.3

By a change of index

$$\left(\frac{|z|}{z}\right)^{\lfloor nq\rfloor} K_n(z,w) \left(\frac{|w|}{\bar{w}}\right)^{\lfloor nq\rfloor}$$

$$= \sum_{k=-\lfloor nq\rfloor}^{n-1-\lfloor nq\rfloor} a_{\lfloor nq\rfloor+k}\, z^k \bar{w}^k |z|^{\lfloor nq\rfloor} |w|^{\lfloor nq\rfloor} e^{-(n+1)V(z)} e^{-(n+1)V(w)}.$$

Proof Annulus case, p.3

By a change of index

$$\left(\frac{|z|}{z}\right)^{\lfloor nq\rfloor} K_n(z,w)\left(\frac{|w|}{\bar{w}}\right)^{\lfloor nq\rfloor}$$
$$= \sum_{k=-\lfloor nq\rfloor}^{n-1-\lfloor nq\rfloor} a_{\lfloor nq\rfloor+k}\, z^k \bar{w}^k |z|^{\lfloor nq\rfloor}|w|^{\lfloor nq\rfloor} e^{-(n+1)V(z)} e^{-(n+1)V(w)}.$$

$$\lim_{n\to\infty}\left(\frac{|z|}{z}\right)^{\lfloor nq\rfloor} K_n(z,w)\left(\frac{|w|}{\bar{w}}\right)^{\lfloor nq\rfloor} = \sum_{k\in\mathbb{Z}} b_k z^k \bar{w}^k |z|^{-\gamma}|w|^{-\gamma}$$

where

$$b_k = \lim_{n\to\infty} a_{\lfloor nq\rfloor+k} = \left(\int_{\mathbb{A}_R} |z|^{2k}|z|^{-2\gamma}\mathrm{d}\ell_{\mathbb{C}}(z)\right)^{-1}.$$

Proof of the behavior at the unit circle

Proof of the behavior at the unit circle, p.1

Study

$$\rho_n(r) = \frac{1}{n^2} \sum_{k=0}^{n-1} a_k \left(1 + \frac{r}{n}\right)^{\left(\frac{k}{n}\right)n}$$

where

$$a_k = \left(\int_{\mathbb{C}} |z|^{2k} e^{-2(n+1)V(z)} \mathrm{d}\ell_{\mathbb{C}}(z)\right)^{-1}.$$

Proof of the behavior at the unit circle, p.1

Study

$$\rho_n(r) = \frac{1}{n^2} \sum_{k=0}^{n-1} a_k \left(1 + \frac{r}{n}\right)^{\left(\frac{k}{n}\right)n}$$

where

$$a_k = \left(\int_{\mathbb{C}} |z|^{2k} e^{-2(n+1)V(z)} \mathrm{d}\ell_{\mathbb{C}}(z) \right)^{-1}.$$

Define $t^*(t) = \lfloor tn \rfloor / n$.

Proof of the behavior at the unit circle, p.1

Study

$$\rho_n(r) = \frac{1}{n^2} \sum_{k=0}^{n-1} a_k \left(1 + \frac{r}{n}\right)^{\left(\frac{k}{n}\right)n}$$

where

$$a_k = \left(\int_{\mathbb{C}} |z|^{2k} e^{-2(n+1)V(z)} \mathrm{d}\ell_{\mathbb{C}}(z) \right)^{-1}.$$

Define $t^*(t) = \lfloor tn \rfloor / n$. Then

$$\rho(r) = \int_0^1 \frac{a_{t^*n}}{n} \left(1 + \frac{r}{n}\right)^{t^*n} \mathrm{d}t.$$

Proof of the behavior at the unit circle, p.2

Since

$$\lim_{n\to\infty} \frac{a_{t^* n}}{n} = \frac{1}{\pi} \frac{(Q-t)(t-q)}{Q-q} 1_{(q,Q)}(t)$$

Proof of the behavior at the unit circle, p.2

Since

$$\lim_{n\to\infty} \frac{a_{t^*n}}{n} = \frac{1}{\pi}\frac{(Q-t)(t-q)}{Q-q}1_{(q,Q)}(t) \quad \text{and} \quad \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{t^*n} = e^{rt}$$

Proof of the behavior at the unit circle, p.2

Since

$$\lim_{n\to\infty} \frac{a_{t^* n}}{n} = \frac{1}{\pi} \frac{(Q-t)(t-q)}{Q-q} 1_{(q,Q)}(t) \quad \text{and} \quad \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{t^* n} = e^{rt}$$

we have

$$\lim_{n\to\infty} \rho_n(r) = \frac{1}{\pi} \int_q^Q \frac{(Q-t)(t-q)}{Q-q} e^{rt} \mathrm{d}t.$$

Proof of the behavior at the unit circle, p.2

Since

$$\lim_{n\to\infty} \frac{a_{t^*n}}{n} = \frac{1}{\pi}\frac{(Q-t)(t-q)}{Q-q}1_{(q,Q)}(t) \quad \text{and} \quad \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{t^*n} = e^{rt}$$

we have

$$\lim_{n\to\infty} \rho_n(r) = \frac{1}{\pi}\int_q^Q \frac{(Q-t)(t-q)}{Q-q}e^{rt}\mathrm{d}t.$$

By Montel's theorem, the same for $r \in \mathbb{C}$.

Proof of the behavior at the unit circle, p.2

Since

$$\lim_{n\to\infty} \frac{a_{t^*n}}{n} = \frac{1}{\pi} \frac{(Q-t)(t-q)}{Q-q} 1_{(q,Q)}(t) \quad \text{and} \quad \lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{t^*n} = e^{rt}$$

we have

$$\lim_{n\to\infty} \rho_n(r) = \frac{1}{\pi} \int_q^Q \frac{(Q-t)(t-q)}{Q-q} e^{rt} \mathrm{d}t.$$

By Montel's theorem, the same for $r \in \mathbb{C}$. Then, choose

$$r_n = z + \bar{w} + \frac{z\bar{w}}{n}$$

that satisfies

$$\lim_{n\to\infty} r_n = z + \bar{w} \quad \text{uniformly on compact sets.}$$