Exercise Sheet 1. First properties and examples

By our definition, a Riemann surface is always connected. We denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

1 Some basic properties

Let X and Y be Riemann surfaces.

Exercise 1 (Removable singularity)**.** Let $x \in X$ and let $f : X \setminus \{x\} \to \mathbb{D}$ be holomorphic. Show that there exists a unique holomorphic function $\tilde{f} : X \to \mathbb{D}$ that extends f.

Exercise 2. Suppose X and Y are compact and consider $\{x_1, \ldots, x_m\} \subset X$ and $\{y_1, \ldots, y_n\} \subset Y$. Show that every biholomorphism between $X \setminus \{x_1, \ldots, x_m\}$ and $Y \setminus \{y_1, \ldots, y_n\}$ can be extended to a biholomorphism between X and Y and, in particular, $m = n$.

Exercise 3. Let S be a topological surface and let $p_1, \ldots, p_n \in S$. Suppose that $S \setminus \{p_1, \ldots, p_n\}$ is endowed with a holomorphic atlas. Show that there exists *at most* one holomorphic maximal atlas over S compatible with the atlas of $S \setminus \{p_1, \ldots, p_n\}$. Give an example where such an atlas over S does not exist.

Exercise 4 (The identity theorem)**.** Let $f, g : X \to Y$ be two holomorphic mappings. Show that if the set $\{x \in X : f(x) = g(x)\}$ has an accumulation point, then $f = g$.

Exercise 5 (Open mapping theorem)**.** Let $f : X \to Y$ be a holomorphic non-constant mapping. Show that f is open.

Exercise 6. Let $f : X \to Y$ be a holomorphic non-constant mapping. Show that if X is compact, then f is surjective and Y is compact. Deduce that, if X is compact, then there do not exist non-constant holomorphic functions $f : X \to \mathbb{C}$ and, in particular, that we cannot embed a compact Riemann surface into $\mathbb{C}^n$ (no naive Whitney embedding theorem).

Exercise 7. Justify the following assertion:

$$\mathcal{M}(X) \simeq \{\text{holomorphic mappings } f : X \to \mathbb{C}P^1\} \setminus \{\infty\}.$$

Would it make sense to identify a quotient of holomorphic functions with a mapping to $\mathbb{C}P^1$ if our complex manifold has higher dimension?

Exercise 8. Suppose that X and Y are compact. Consider a holomorphic mapping $f : X \to Y$, a point $y \in Y$ and suppose that $f^{-1}(y) = \{x_1, \ldots, x_k\} \subset X$. Prove the existence of neighborhoods $D, D_1, \ldots, D_k$ of $x, y_1, \ldots, y_k$ and biholomorphisms $\psi : \mathbb{D} \to D$ and $\psi_i : \mathbb{D} \to D_i$ such that

- $\psi(0) = x$,
- $\psi_i(0) = y_i$ for every $i \in \{1, \ldots, k\}$,
- $f^{-1}(D) = D_1 \sqcup \cdots \sqcup D_k$ and
- $f(\psi_i(z)) = \psi(z^{n_i})$ for every $z \in \mathbb{D}$ (with $n_i \geq 1$).

Exercise 9 (Simple version of Radó's theorem)**.** Let S be a compact topological space which is locally homeomorphic to $\mathbb{C}$. Show that S is second countable.

2 Two standard examples

Exercise 10. Consider the sphere S^2 as a topological space and $p_0 \in S^2$. Given a homeomorphism $\varphi : S^2 \setminus \{p_0\} \to \mathbb{C}$ show that S^2 admits a unique holomorphic maximal atlas such that φ is a chart, and let us denote the sphere endowed with such maximal atlas by S^2_φ. Show that if $q_0 \in S^2$ and $\psi : S^2 \setminus \{q_0\} \to \mathbb{C}$ is another homeomorphism, then S^2_φ is biholomorphic to S^2_ψ.

Exercise 11. Let $\Gamma \subset \mathbb{C}$ be a discrete subgroup.

1. Show that Γ is isomorphic to $\{0\}$, $\mathbb{Z}$, or $\mathbb{Z}^2$.
2. Show that there exists a unique maximal atlas over $\mathbb{C}/\Gamma$ such that $\mathbb{C} \to \mathbb{C}/\Gamma$ is holomorphic.

3 Hyperelliptic curves over $\mathbb{C}$ (first try)

Let $Q : \mathbb{C}^3 \to \mathbb{C}$ be a non-degenerated quadratic form and let

$$S = \{[z_1, z_2, z_3] \in \mathbb{C}P^2 : Q(z_1, z_2, z_3) = 0\} \subset \mathbb{C}P^2.$$

Exercise 12 (Quadric)**.** Show that S is a Riemann surface (1-dimensional connected complex submanifold[1] of $\mathbb{C}P^2$) biholomorphic to $\mathbb{C}P^1$.

Consider a monic polynomial $P \in \mathbb{C}[z]$ having pairwise distinct roots $a_1, \dots, a_k$, with $k \geq 1$, i.e., $P(z) = (z - a_1) \dots (z - a_k)$. Let us consider the subspace

$$X = \{(z, w) \in \mathbb{C}^2 : w^2 = P(z)\}.$$

Exercise 13. Show that X is a Riemann surface.

Consider $\mathbb{C}^2 \hookrightarrow \mathbb{C}P^2$ where (z, w) becomes $[z, w, 1]$ and take the closure $\bar{X}$ of X in $\mathbb{C}P^2$. Equivalently, considering the homogenization of the polynomial $w^2 - P(z)$, we have, if $k \geq 2$,

$$\bar{X} = \{[z, w, t] \in \mathbb{C}P^2 : w^2 t^{k-2} = (z - a_1 t) \dots (z - a_k t)\}.$$

Exercise 14. Show that $\bar{X}$ is a Riemann surface if and only if $k \leq 3$. What is $\bar{X}$ for $k = 1$ or 2?

Later in the course, we will consider a "better" compactification of X by a finite set.

[1]For a complex manifold X of dimension m we will say that a subset $Y \subset X$ is a complex submanifold of dimension n if we may find a chart $\varphi : U \subset X \to \mathbb{C}^m$ around each point of Y such that $\varphi(Y \cap U) = (\mathbb{C}^n \times \{0\}) \cap \varphi(U)$.