First notions of covering spaces

Here we recall some notions of covering spaces assuming the basic notions (essentially just definitions) about the fundamental group are known. It is intended to be more guided than the "*Notions of complex analysis via exercises*" but I am afraid it may contain much more mistakes[1]. So, your help in spotting them is very welcome![2] This document contains only some very basic properties of covering spaces, but some subjects may be added later if needed for the Riemann surfaces course. There is an appendix on two of the classical definitions of proper maps and how they are related (just for the ease of mind).

Caution: These exercises are not at all needed for the course! The few results that are needed will be discussed in class. So, this document is meant to help someone that wants to quickly learn some of the most basic properties by trying to prove them by themselves.

Most of these exercises may be found as theorems and propositions of "*Fundamental groups and covering spaces*" by Elon Lages Lima which I recommend to read without reading the proofs when possible. Another good reference is "*Algebraic Topology*" by Allen Hatcher.

Definition 1. Let X and Y be two topological spaces and let $p : X \to Y$ be a continuous map. We say that p is a *covering map* and that X is a covering space of Y if, for every $y \in Y$, there exists an open neighborhood V of y such that we can write $p^{-1}(V) = \sqcup_{\lambda\in\Lambda} U_\lambda$ with $(U_\lambda)_{\lambda\in\Lambda}$ open and pairwise disjoint, and such that $f|_{U_\lambda} : U_\lambda \to V$ is a homeomorphism. We shall call such V a *trivializing neighborhood* of p.

If the reader is more familiar with fiber bundles, the following description may be interesting.

Exercise 2. Let X and Y be two topological spaces and let $p : X \to Y$ be a continuous map. Show that the following two properties are equivalent.

- p is a covering map
- Y can be covered by a family $(V_\lambda)_{\lambda\in\Lambda}$ of open sets such that for each λ there exists a discrete space D_λ and a homeomorphism $\varphi_\lambda : D_\lambda \times V_\lambda \to p^{-1}(V_\lambda)$ satisfying $p \circ \varphi_\lambda = p_2$, where $p_2 : D_\lambda \times V_\lambda \to V_\lambda$ is the projection onto the second coordinate.

The first main result is the following.

Proposition 3 (Path lifting property)**.** *Let $p : X \to Y$ be a covering map and let $\gamma : [0,1] \to Y$ be a continuous path. For any $x_0 \in X$ such that $p(x_0) = \gamma(0)$, there exists a unique $\widetilde{\gamma} : [0,1] \to X$ such that $\gamma = p \circ \widetilde{\gamma}$. We say that $\widetilde{\gamma}$ is a* lift *of γ by p.*

We will prove this in two steps: **existence** and **uniqueness**.

Exercise 4. Fix a covering map $p : X \to Y$, a continuous path $\gamma : [0,t] \to Y$ and $x_0 \in p^{-1}(\gamma(0))$. Consider $A = \{t \in [0,1] :$ there exists $\widetilde{\gamma} : [0,t] \to X$ satisfying $\gamma|_{[0,t]} = p \circ \widetilde{\gamma}$ and $\widetilde{\gamma}(0) = x_0\}$.

- Show that A is open.
  Hint: Use $f|_{U_\lambda} : U_\lambda \to V$ from the definition to extend $\widetilde{\gamma}$ from $[0,t]$ to $[0,t+\varepsilon]$.
- Show that A is closed.
  Hint: Use $f|_{U_\lambda} : U_\lambda \to V$ from the definition to extend $\widetilde{\gamma}$ from $[0,t-\varepsilon]$ to $[0,t]$.

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[1] I will try to give a careful look at these notes but I cannot make any promises.

[2] You may write to david.garcia-zelada@sorbonne-universite.fr for any remarks and questions.

- What can we conclude?
- Show the existence of $\widetilde{\gamma} : [0,1] \to X$ in a more straightforward way by dividing $[0,1]$ in small intervals whose image is contained in trivializing neighborhoods of p.

We may notice that in the previous exercise we can replace p by any fiber bundle.

Exercise 5. Let $f : X \to Y$ be a continuous map, let E be a topological space and consider two continuous maps $g_1, g_2 : E \to X$ satisfying $f \circ g_1 = f \circ g_2$. Define $A = \{e \in E : g_1(e) = g_2(e)\}$

- Suppose that X can be covered by open sets such that f restricted to each one of them is injective. Show that A is open.
- Suppose that any pair of points $x_1 \neq x_2$ in X such that $f(x_1) = f(x_2)$ admit disjoint neighborhoods. Show that A is closed.
- What can we conclude if f is a covering map and E is connected?

Now, let us try to lift homotopies between paths.

Exercise 6. Let $p : X \to Y$ be a covering map and $H : [0,1] \times [0,1] \to Y$ a continuous map. Take $x_0 \in \pi^{-1}(H(0,0))$.

1. Show that there exists $n \in \mathbb{N}$ and a family $(V_{i,j})_{i,j \in \{1,\dots,n\}}$ of trivializing neighborhoods such that, if we define the squares $Q_{i,j} = [\frac{i-1}{n}, \frac{i}{n}] \times [\frac{j-1}{n}, \frac{j}{n}]$, we have $H(Q_{i,j}) \subset V_{i,j}$.
2. Consider a lexicographic order in $\{1,\dots,n\}^2$, i.e., $(i,j) \leq (i',j')$ if and only if $j < j'$ or $j = j'$ and $i \leq i'$. For each $(i,j) \in \{1,\dots,n\}^2$ define $C_{i,j} = \cup_{(i',j') \leq (i,j)} Q_{i',j'}$. Consider the maximum μ of the set of $(i,j) \in \{1,\dots,n\}^2$ such that there exists a continuous map $\widetilde{H} : C_{i,j} \to Y$ satisfying $p \circ \widetilde{H}|_{C_{i,j}} = H$ and $\widetilde{H}(0,0) = x_0$. Show that $\mu = (n,n)$.

Exercise 7 (Homotopy Lifting Property for paths)**.** Let $E = [0,1]$, let $p : X \to Y$ be a covering map and $H : E \times [0,1] \to Y$ a continuous map. Suppose that $\gamma : E \to X$ is a continuous map satisfying $p \circ \gamma = H(\cdot, 0)$. Show that there exists a unique continuous map $\widetilde{H} : E \times [0,1] \to X$ that satisfies $p \circ \widetilde{H} = H$ and $\widetilde{H}(\cdot,0) = \gamma$.

The reason why we distinguished E from the second coordinate $[0,1]$ is because the general homotopy lifting property with a general topological space E also holds. It may be proved following the same ideas as the ones in Exercise 6. This can be found, for instance, in page 30 of Allen Hatcher's "*Algebraic Topology*". Notice that Proposition 3 and the result of Exercise 7 would be consequences of this general homotopy lifting property.

Another approach is to first prove that the "path lifting map" $X \times_Y C([0,1], Y) \to C([0,1], Y)$ is continuous. Here $X \times_Y C([0,1], Y) = \{(x, \gamma) \in X \times C([0,1], Y) : p(x) = \gamma(0)\}$ and the map considered takes (x, γ) to the unique continuous path $\widetilde{\gamma} : [0,1] \to X$ that satisfies $\widetilde{\gamma}(0) = x$ and $p \circ \widetilde{\gamma} = \gamma$. The data (H, γ) can be thought of as a path $t \mapsto (\gamma(t), H(t, \cdot)) \in X \times_Y C([0,1], Y)$ so that we just compose by the "path lifting map" to obtain $\widetilde{H}$. The same idea allows us to quickly show the general homotopy lifting property. This can be found in Proposition 6.10 of "*Fundamental groups and covering spaces*" by Elon Lages Lima.

Exercise 8 (Lifting of maps)**.** Let $p : X \to Y$ be a covering map and let E be topological space with a chosen point $e_0 \in E$. Take a continuous map $f : E \to Y$ and a point $x_0 \in X$ satisfying $f(e_0) = p(x_0)$ and suppose that $f_*(\pi_1(E, e_0)) \subset p_*(\pi_1(X, x_0))$.

1. Let $\gamma_1, \gamma_2 : [0,1] \to E$ be two continuous paths that begin at e_0 and coincide at 1. Show that then the lifts of $f \circ \gamma_1$ and $f \circ \gamma_2$ that begin at x_0 also coincide at 1.

2. Suppose that E is path-connected. Show that there exists a unique map (not asking if it is continuous) $\widetilde{f} : E \to X$ satisfying the following properties.

   - $p \circ \widetilde{f} = f$,
   - $\widetilde{f}(e_0) = x_0$ and
   - for any continuous path $\gamma : [0,1] \to E$ such that $\gamma(0) = e_0$, the map $\widetilde{f} \circ \gamma$ is continuous.

3. Suppose an open set W that is path-connected and such that $f(W)$ is contained in a trivializing neighborhood of p. Show that $\widetilde{f}$ is continuous in W.

4. Show that, if E is connected and locally path-connected, there exists a unique continuous map $\widetilde{f} : E \to X$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(e_0) = x_0$.

5. Show that the condition $f_*(\pi_1(E, e_0)) \subset p_*(\pi_1(X, x_0))$ is actually necessary for the existence of such $\widetilde{f}$ (without any assumption on E).

Exercise 9. Suppose that Y is locally path connected and let $p_1 : X_1 \to Y$ and $p_2 : X_2 \to Y$ be two covering maps with X_1 and X_2 connected. Let $x_1 \in X_1$ and $x_2 \in X_2$ satisfying that $p_1(x_1) = p_2(x_2)$. Show that the following conditions are equivalent.

- There exists a homeomorphism $F : X_1 \to X_2$ such that $p_2 \circ F = p_1$ and $F(x_1) = x_2$.
- $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_2))$.

Let us look at some sufficient conditions for a local homeomorphism to be a covering map.

Definition 10. We say that a map $f : X \to Y$ is a local homeomorphism if every $x_0 \in X$ admits an open neighborhood U such that $f(U)$ is open and $f|_U : U \to f(U)$ is a homeomorphism.

Exercise 11. Let X be Hausdorff and let $f : X \to Y$ be a local homeomorphism. Prove that if the cardinal of $f^{-1}(y)$ is finite and does not depend on $y \in Y$, then f is closed.

Exercise 12. Let X be Hausdorff and let $f : X \to Y$ be a local homeomorphism. Suppose that $f^{-1}(y)$ is finite (or, equivalently since f is a local homeomorphism, compact) and that f is closed. Show that f is a covering map.

Appendix: About proper maps

Exercise 13 (Closed vs "continuity" of the pullback)**.** Let $f : X \to Y$ be a map (not necessarily continuous). Show that the following assertions are equivalent.

- For every closed subset $C \subset X$, the set $f(C) \subset Y$ is closed.
- For every $y \in Y$ and every open neighborhood U of $f^{-1}(y)$, there exists an open neighborhood V of y such that $f^{-1}(V) \subset U$.
- For every $C \subset Y$ and every open neighborhood U of $f^{-1}(C)$, there exists an open neighborhood V of C such that $f^{-1}(V) \subset U$.

When these equivalent conditions hold for f, we say that f is a *closed map*.

The third assertion can be interpreted as saying that the map $f^{-1} : 2^Y \to 2^X$ is continuous[3].

Exercise 14. Let $f : X \to Y$ be a closed map such that $f^{-1}(y)$ is compact for every $y \in Y$. Show that $f^{-1}(K)$ is compact for every compact $K \subset Y$.
Hint: Notice that an open cover for $f^{-1}(K)$ is a cover of each fiber $f^{-1}(y)$ for $y \in K$ and use the second (equivalent) definition of a closed map to find a covering of K.

Exercise 15. Let $f : X \to Y$ be a function for which the preimage and the image of any compact set is a compact set. Show that, for every closed set $C \subset X$ and for every compact set $K \subset Y$, the set $f(C) \cap K$ is compact.

Exercise 16. Suppose that Y is Hausdorff and locally compact. Show that a subset $C \subset Y$ is closed if and only if $K \cap C$ is compact for every compact set $K \subset Y$.
Hint: If C is closed and $y \in Y \setminus f(C)$, look at a compact neighborhood of y.

Exercise 17. Suppose that Y is Hausdorff and locally compact and let $f : X \to Y$ be a function for which the preimage and the image of any compact set is a compact set. Show that f is closed.

Exercise 18 (Proper map)**.** Suppose that Y is Hausdorff and locally compact. Let $f : X \to Y$ be a continuous map and show that the following two properties are equivalent.

- $f^{-1}(K)$ is compact for every compact $K \subset Y$.
- f is closed and $f^{-1}(y)$ is compact for every $y \in Y$.

When these equivalent assertions hold for f, we say that f is proper (if Y is not Hausdorff or locally compact, we would have to choose one definition, but in our context there is no problem).

[3] In a slightly more precise way, we may define the continuity of a relation $R \subset X \times Y$ and notice that a function is continuous if and only if it is continuous as a relation and that it is closed if and only if the inverse relation is continuous