Exercise Sheet 2. Divisors

Let X be a (connected and) compact Riemann surface.

- $\mathrm{Div}(X)$ = free abelian group generated by X. An element of $\mathrm{Div}(X)$ can be written as $\sum_{i=1}^k n_i[p_i]$ with $n_i \in \mathbb{Z}$ and $p_i \in X$, and the elements of $\mathrm{Div}(X)$ are called *divisors*[1].
- The morphism $\mathrm{div} : \mathcal{M}(X)^* \mapsto \mathrm{Div}(X)$ is given by
$$\mathrm{div} f = \sum_{p \in X} \mathrm{ord}_p f\, [p].$$
- A divisor $\mathrm{div} f$ is called a *principal divisor* and $\mathrm{Princ}(X) = \{\mathrm{div} f : f \in \mathcal{M}(X)^*\}$.
- Two divisors are *linearly equivalent* if their difference is a principal divisor.
- Define $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ by $\deg(\sum_{i=1}^k n_i[p_i]) = \sum_{i=1}^k n_i$ and its kernel $\mathrm{Div}^0(X)$.
- We say $D = \sum_{i=1}^k n_i[p_i] \in \mathrm{Div}(X)$ is *effective* and write $D \geq 0$ if $n_i \geq 0$ for every i.
- For two divisors $D_1, D_2 \in \mathrm{Div}(X)$, we write $D_1 \geq D_2$ if $D_1 - D_2$ is effective.
- For $D \in \mathrm{Div}(X)$, we consider the vector space
$$L(D) = \{f \in \mathcal{M}(X) : f = 0 \text{ or } \mathrm{div} f + D \geq 0\}.$$
- We use the notation $\ell(D) = \dim(L(D))$.

Exercise 1. Show that if D_1 and D_2 are linearly equivalent then $\deg(D_1) = \deg(D_2)$.

Exercise 2. Let $f_1, f_2 \in \mathcal{M}(X)^*$. Show that if $\mathrm{div} f_1 = \mathrm{div} f_2$ then $f_1 = \alpha f_2$ with $\alpha \in \mathbb{C}^*$.

Exercise 3. Let $f \in \mathcal{M}(X)^*$. Show that $L(\mathrm{div} f)$ is generated by $1/f$.

Exercise 4. Show that $\mathrm{Div}(X) \simeq \mathbb{Z} \times \mathrm{Div}^0(X)$.

Exercise 5. Show that if D_1 and D_2 are linearly equivalent then $\ell(D_1) = \ell(D_2)$.

Exercise 6. Let $D \in \mathrm{Div}(X)$ be an effective divisor. Show that $\ell(D) \leq \deg(D) + 1$.

Exercise 7. Show that if $\deg(D) < 0$ then $L(D) = 0$.

Exercise 8. Show that $\ell(D) > 0$ if and only if there exists $\tilde{D} \sim D$ such that $\tilde{D} \geq 0$.

Exercise 9. Suppose that $E \in \mathrm{Div}(X)$ satisfies $\ell(D - E) > 0$. Show that there exists $\tilde{D} \in \mathrm{Div}(X)$ such that $\tilde{D} \sim D$ and $\tilde{D} \geq E$.

Exercise 10. Let $D \in \mathrm{Div}^0(X)$. Show that $D \in \mathrm{Princ}(X)$ if and only if $\ell(D) \geq 1$.

Exercise 11. Let $D \in \mathrm{Div}^0(X)$. Show that $\ell(D) = \ell(-D)$.

[1] The group $\mathrm{Div}(X)$ has a natural topological group structure by identifying $\sum_{i=1}^k n_i[p_i]$ with the measure $\sum_{i=1}^k n_i \delta_{p_i}$ and considering the weak topology on the space of finite measures..

Exercise 12 (Sphere case)**.** Consider $X = \mathbb{C} \cup \{\infty\}$.

1. Is $f : X \setminus \{\infty\} \to \mathbb{C}$ given by $f(z) = z$ meromorphic over X? What is its divisor?
2. Show that every holomorphic function over X is constant by looking at it under the two charts (without using Liouville's theorem).
3. Show that $\text{Princ}(X) = \text{Div}^0(X)$.
4. Describe $L(k[\infty])$ explicitly for $k \geq 0$. What do we obtain for $k < 0$?
5. Let $D \in \text{Div}(X)$. Find $\ell(D)$.
6. Give a (more or less) explicit description of $L(D)$ for $D \in \text{Div}(X)$.

Exercise 13 (Torus case)**.** Consider an elliptic curve $X = \mathbb{C}/\Gamma$.

1. Show that $\text{Div}^0(X)/\text{Princ}(X)$ is isomorphic to[2] $\mathbb{C}/\Gamma$. Deduce that $\text{Div}(X)/\text{Princ}(X)$ is isomorphic to $\mathbb{Z} \times (\mathbb{C}/\Gamma)$.
2. Suppose that $D \in \text{Div}(X)$ satisfies $\deg(D) \geq 1$. Show that there exists $p \in X$ such that D is linearly equivalent to $\deg(D)[p]$.
3. Let $D \in \text{Div}(X)$ satisfy $\deg(D) \geq 1$. Show that $\ell(D) = \deg(D)$.

Exercise 14. Let $k \geq 1$ and let $P = (z - a_1) \dots (z - a_k)$ be a monic polynomial with simple roots. Let us consider the Riemann surface

$$X = \{(z, w) \in \mathbb{C}^2 : w^2 = P(z)\}.$$

Take the functions w and z obtained by restricting the coordinates w and z of $\mathbb{C}^2$ to X.

1. Are the functions w and z holomorphic over X ? Justify your answer using the definition.
2. Find $\text{div}\, w$.
3. Find $\text{div} z$.

Suppose that we have a **compact** Riemann surface S containing X such that $S \setminus X$ has a finite number of points (in particular, X is an open subset of S).

4. Can we extend w and z to S? Holomorphically? Meromorphically?
5. Show that $S \setminus X$ has at most 2 points.

[2] We can also show that they are isomorphic as topological groups.