Edge fluctuations for random normal matrix ensembles

Let me define the one I consider a geometric and nice-looking case. Consider a probability measure ν of finite logarithmic energy on $\mathbb{C}$ and define

$$V^\nu(z) = \int_{\mathbb{C}} \log|z-w| \mathrm{d}\nu(w).$$

Consider n particles of charge $q_n = \frac{1}{n+1}$ interacting with the charge $-\nu$ so that

$$H_n^\nu(z_1,\dots,z_n) = -q_n^2 \sum_{i<j} \log|z_i - z_j| + q_n \sum_{i=1}^n V^\nu(z_i).$$

is the total potential energy of this system of particles. Consider the system at inverse temperature $\beta_n = 2(n+1)^2$, i.e., consider the probability measure on $\mathbb{C}^n$

$$\begin{aligned}\mathrm{d}\mathbb{P}_n^\nu(z_1,\dots,z_n) &= \frac{1}{\mathcal{Z}} \exp(-\beta_n H_n^\nu(z_1,\dots,z_n)) \mathrm{d}\ell_{\mathbb{C}^n}(z_1,\dots,z_n) \\ &= \frac{1}{\mathcal{Z}} \prod_{i<j} |z_i - z_j|^2 e^{-2(n+1)\sum_{i=1}^n V^\nu(z_i)} \mathrm{d}\ell_{\mathbb{C}^n}(z_1,\dots,z_n).\end{aligned}$$

The reason why it may be nice to choose a charge $\frac{1}{n+1}$ for each particle is because in this case the map $\nu \mapsto \mathbb{P}_n^\nu$ is invariant under conformal transformations[1] (a.k.a. equivariant map).

On the other hand, the reason why it is nice to have the exponent 2 in $\prod_{i<j}|z_i - z_j|^2$ is that points chosen according to $\mathbb{P}_n^\nu(z_1,\dots,z_n)$ form a determinantal point process. More precisely, the information of $\mathbb{P}_n^\nu$ is contained in the orthogonal projection

$$L^2(\mathbb{C}, e^{-2(n+1)V^\nu}\mathrm{d}\ell_{\mathbb{C}}) \twoheadrightarrow \mathrm{Pol}_{\leq n-1}(\mathbb{C}),$$

where $\mathrm{Pol}_{\leq n-1}(\mathbb{C})$ denotes the space of polynomials of degree less or equal than $n-1$. Since V^ν is logarithmic at infinity, $\mathrm{Pol}_{\leq n-1}(\mathbb{C})$ is exactly the set $L^2_{\mathrm{Hol}}(\mathbb{C}, e^{-2(n+1)V^\nu}\mathrm{d}\ell_{\mathbb{C}})$ of L^2 holomorphic functions. We will denote this point process by $\mathcal{C}_n^\nu$.

On the other hand, if we start with a general positive measure μ and take a function U such that $\Delta U = \mu$, we could consider the determinantal point process associated to the projection

$$L^2(\mathbb{C}, e^{-2U}\mathrm{d}\ell_{\mathbb{C}}) \twoheadrightarrow L^2_{\mathrm{Hol}}(\mathbb{C}, e^{-2U}\mathrm{d}\ell_{\mathbb{C}}).$$

It can be seen that this only depends on[2] μ and not on the precise U we have chosen (as long as $\Delta U = \mu$). We denote this point process by $\mathcal{C}^\mu$.

One of the main philosophies I wanted to convey in this article is that "if μ_n converges (in some sense) to μ, then $\mathcal{C}^{\mu_n}$ should converge to $\mathcal{C}^\mu$". If T_{λ,x_0} denotes the map $x \mapsto \lambda(x-x_0)$, *Edge fluctuations for random normal matrix ensembles* shows the following for a family of radial cases.

$$\text{If} \quad nT_{\lambda_n,x_0}\nu \xrightarrow[n\to\infty]{} \mu \quad \text{then} \quad T_{\lambda_n,x_0}\mathcal{C}_n^\nu \xrightarrow[n\to\infty]{} \mathcal{C}^\mu.$$

[1]Moreover, this invariance can be nicely explained from the point of view of holomorphic Hermitian line bundles on the sphere.

[2]This is because if U is harmonic on $\mathbb{C}$, there exists a holomorphic function f on $\mathbb{C}$ such that $|f|^2 = e^{-2U}$.

Remark 1. *Another goal of this article is to convey the idea that the dominated convergence theorem and a few properties of holomorphic functions are the only tools necessary to deal with a bunch of questions for radial determinantal Coulomb gases.*

Remark 2. *For the case $\nu(\mathbb{C}) > 1$, the inclusion* $\mathrm{Pol}_{\leq n-1}(\mathbb{C}) \subset L^2_{\mathrm{Hol}}(\mathbb{C}, e^{-2(n+1)V^\nu}\mathrm{d}\ell_{\mathbb{C}})$ *is strict so that convergence towards $L^2 \twoheadrightarrow X \subset L^2_{\mathrm{Hol}}$ for some closed subspace $X \neq L^2_{\mathrm{Hol}}$ is expected, and proven for a family of radial determinantal Coulomb gases in* Edge fluctuations for random normal matrix ensembles *that generalize the finite Ginibre point process.*

Remark 3. *The philosophy in red above is tied to the fact that $\mathcal{C}^\mu$ does not depend on U as long as $\Delta U = \mu$. Nevertheless, we could have that μ_n converges to infinity outside some open set A and $\mu_n|_A$ converges to some μ. In this case, we would have a family of possible limits (limits of subsequences) for $\mathcal{C}^{\mu_n}|_A$ indexed by a set that depends on the homology of A. This is explored for some cases in* Universality for outliers in weakly confined Coulomb-type systems *and in* Extremal particles of two-dimensional Coulomb gases and random polynomials on a positive background.

Remark 4. *In* Edge fluctuations for random normal matrix ensembles, *fluctuations of the maxima are also (even primarily) discussed. The proofs follow the same ideas as below, the main point being that, by an argument due to Kostlan, the system of distances to the origin behaves as a system of independent variables and, in the Poisson case, we only need the limit of intensities.*

Idea of the proof

Suppose that ν is radial, that $\nu(\{z \in \mathbb{C} : |z| > 1\}) = 0$ and that, for some $\alpha > 1$,

$$\nu(\{z \in \mathbb{C} : |z| > r\}) = \alpha(1-r)^{\alpha-1} + o(1-r)^{\alpha-1} \text{ as } r \to 1^-.$$

Then, modulo an additive constant, $V^\nu(z) = V(|z|)$ for some $V : [0,\infty) \to \mathbb{R}$ satisfying

- $V(r) = \log r$ for every $r \geq 1$,
- $V(r) > \log r$ for every $r < 1$ and
- $V(r) = \log r + (1-r)^\alpha + o(1-r)^\alpha$ as $r \to 1^-$.

Now it can be seen, for instance, that if $x \in \mathbb{R}$, the intensity function of the determinantal point process $T_{n^{1/\alpha},1}\mathcal{C}^\nu_n$ at x is (for n large enough)

$$\frac{1}{n^{2/\alpha}} \sum_{k=0}^{n-1} a_k^{(n)} \left(1 + \frac{x}{n^{1/\alpha}}\right)^{2k} e^{-2(n+1)V\left(1+\frac{x}{n^{1/\alpha}}\right)}, \text{ where } a_k^{(n)} = \left(2\pi \int_0^\infty r^{2k+1} e^{-2(n+1)V(r)}\mathrm{d}r\right)^{-1}.$$

To find the limit we write

$$\frac{1}{n^{2/\alpha}} \sum_{k=0}^{n-1} a_k^{(n)} \left(1 + \frac{x}{n^{1/\alpha}}\right)^{2k} e^{-2(n+1)V\left(1+\frac{x}{n^{1/\alpha}}\right)}$$
$$= \frac{1}{n^{2/\alpha}} \sum_{k=0}^{n-1} a_k^{(n)} \left(1 + \frac{x}{n^{1/\alpha}}\right)^{2(k-n-1)} e^{-2(n+1)\left(V\left(1+\frac{x}{n^{1/\alpha}}\right) - \log\left(1+\frac{x}{n^{1/\alpha}}\right)\right)}.$$

By the behavior of $V(r) - \log r$ near $r = 1$ we have

$$\lim_{n\to\infty} (n+1)\left(V\left(1+\frac{x}{n^{1/\alpha}}\right) - \log\left(1+\frac{x}{n^{1/\alpha}}\right)\right) = |\min(x,0)|^{\alpha} =: U(x).$$

On the other hand, the sum part can be seen as a Riemann sum and we can use the dominated convergence theorem to finally obtain

$$\lim_{n\to\infty} \frac{1}{n^{2/\alpha}} \sum_{k=0}^{n-1} a_k^{(n)} \left(1+\frac{x}{n^{1/\alpha}}\right)^{2k} e^{-2(n+1)V\left(1+\frac{x}{n^{1/\alpha}}\right)} = \frac{e^{-2U(x)}}{2\pi} \int_{-\infty}^{0} \frac{e^{2xt}}{\int_{-\infty}^{\infty} e^{2st} e^{-2U(s)} \mathrm{d}s} \mathrm{d}t =: \rho(x).$$

Notice that, if we extend U as $U(x+iy) = U(x)$, then

$$\Delta U = \alpha(\alpha-1)|\min(x,0)|^{\alpha-2} \mathrm{d}x\mathrm{d}y =: \mu$$

which is the limit of $nT_{n^{1/\alpha},1}\mathcal{C}_n^{\nu}$ due to the properties of ν. Moreover, ρ is the intensity of $\mathcal{C}^{\mu}$.