Exercise Sheet 6. Divisors and Riemann-Roch

Here X will be a compact and connected Riemann surface of genus g and $D \in \mathrm{Div}(X)$.

Exercice 1. Show that if $\deg(D) = 2g-2$ then $\ell(D) = g$ or $\ell(D) = g-1$.

Exercice 2. Show that if $\deg(D) \leq g-1$ and $\ell(D) > 0$ then $\ell(K-D) > 0$.

Exercice 3. Let $p \in X$. Show that $\dim(L(D)/L(D-[p])) = 0$ or 1.

Exercice 4. Show that if there exists $p \in X$ satisfying $\ell([p]) = 2$ then $g = 0$.

Exercice 5. Show that if $\deg(D) \geq 2g$ then $L(D-[p]) \subsetneq L(D)$ for every $p \in X$.

Exercice 6. Suppose that $\ell(D) > 0$. Show that there exists $\tilde{D} \leq D$ such that $L(\tilde{D}) = L(D)$ but $L(\tilde{D}-[p]) \subsetneq L(\tilde{D})$ for every $p \in X$.

Exercice 7. Let $A, B \in \mathrm{Div}(X)$ with disjoint supports [1].

1. Show that $L(A) \cap L(B) \subset \mathbb{C}$.
2. Show that if, in addition, A and B are effective, then $\ell(A+B) \geq \ell(A) + \ell(B) - 1$.
3. Show that this inequality cannot hold in general.

Exercice 8. Suppose that $\ell(D) = n > 0$.

1. Show that there exists a finite set $F \subset X$ such that $L(D-[p]) \subsetneq L(D)$ for every $p \in X \setminus F$.
2. Let $U \subset X$ be a non-empty open set. Show that there exists $\widetilde{D} \in \mathrm{Div}(X)$ effective and $n-1$ pairwise distinct points $p_1, \ldots, p_{n-1} \in U$ such that $D \sim \widetilde{D} + [p_1] + \cdots + [p_{n-1}]$ and
$$\mathbb{C} = L(\widetilde{D}) \subsetneq L(\widetilde{D}+[p_1]) \subsetneq L(\widetilde{D}+[p_1]+[p_2]) \subsetneq \cdots \subsetneq L(\widetilde{D}+[p_1]+\cdots+[p_{n-1}]).$$
3. Show that, for every non-empty and open set $U \subset X$, there exists an effective divisor $\widetilde{D} \sim D$ such that every non-constant $f \in L(\widetilde{D})$ has a pole in U.
4. Let $A, B \in \mathrm{Div}(X)$ be effective. Show that $\ell(A+B) \geq \ell(A) + \ell(B) - 1$.
5. Show that if $\ell(D) > 0$ and $\ell(K-D) > 0$ then $2(\ell(D)-1) \leq \deg(D)$.

Exercice 9. Suppose that $\ell(D) = n > 1$. Take a basis $\{f_1, \ldots, f_n\}$ of $L(D)$ and consider the set $T = \{p \in X : p \text{ is a pole or a zero of at least one of the } f_i\}$. Define the map $\phi_D : X \setminus T \to \mathbb{C}P^{n-1}$ by $\phi_D(p) = [f_1(p), \ldots, f_n(p)]$.

1. Show that ϕ_D can be extended to a holomorphic map over X.
2. Suppose that $\deg(D) \geq 2g+1$. Show that ϕ_D is an injective immersion.
3. If $X = \mathbb{C}P^1$, describe ϕ_D explicitly for $D = k[\infty]$ with $k \geq 1$. In this case, we obtain an inclusion $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^k$. What is the degree of the projective curve $\phi_D(\mathbb{C}P^1)$?
4. Suppose that $\deg(D) \geq 2g+1$ which guarantee that $\phi_D(X)$ is a Riemann surface biholomorphic to X. What is the degree of the projective curve $\phi_D(X)$?
5. Give a more intrinsic description of ϕ_D as a map $X \to \mathbb{P}(L(D)^*)$, where $\mathbb{P}(L(D)^*)$ is the space of complex one-dimensional subspaces of of the dual of $L(D)$.

1. The support of a divisor $\sum_{x \in X} n_x[x]$ is the set $\{x \in X : n_x \neq 0\}$.

Line bundles

When necessary, we will assume known that every holomorphic line bundle admits a meromorphic section and, as a consequence, that $\mathrm{Div}(X)/\mathrm{Princ}(X)$ is identified with the holomorphic line bundles (modulo isomorphism). We will assume also known (or you can guess it as an exercise) the notion of tensor product of holomorphic line bundles and morphisms between holomorphic line bundles (the notion of the dual of a line bundle if necessary).

Exercice 10. Let $\mathcal{L}$ be the line bundle associated to D and fix $p \in X$. Interpret the condition $L(D) = L(D - [p])$ in terms of holomorphic sections of $\mathcal{L}$.

Exercice 11. Let $\mathcal{L}$ be the line bundle associated to D. Show that the holomorphic sections of $\mathcal{L}$ over a connected open $U \subset X$ are naturally identified with $\{f \in \mathcal{M}(U)^* : \mathrm{div} f + D|_U \geq 0\} \cup \{0\}$.

Exercice 12. Let $\mathcal{L}$ be the line bundle associated to D. Show that $\mathcal{L}^*$ is associated to $-D$.

Exercice 13. Let $\mathcal{L}_1$ be the line bundle associated to $D_1 \in \mathrm{Div}(X)$ and $\mathcal{L}_2$ the one associated to $D_2 \in \mathrm{Div}(X)$. Show that $\mathcal{L}_1 \otimes \mathcal{L}_2$ is the line bundle associated to $D_1 + D_2$.

Exercice 14. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two holomorphic line bundles. Show that if there exists a morphism of holomorphic line bundles $\mathcal{L}_1 \to \mathcal{L}_2$ then $\deg \mathcal{L}_1 \leq \deg \mathcal{L}_2$ and that if $\deg \mathcal{L}_1 = \deg \mathcal{L}_2$ then this morphism is an isomorphism.

Exercice 15. Suppose that $g = 1$. Show that there exists exactly four holomorphic line bundles that satisfy $\mathcal{L}^{\otimes_2} = TX$.

Exercice 16. If $g \geq 1$, show that, for every $p \in X$, there exists a holomorphic line bundle where every holomorphic section has a zero at p.

Exercice 17. Is it possible to give a description of ϕ_D from Exercise 9 using line bundles ? What conditions should we ask of D ?

Little reminder. For a divisor $D = \sum_{x \in X} n_x[x]$, we want to construct a holomorphic line bundle $\mathcal{L}$ endowed with a meromorphic section s such that $\mathrm{div} s = D$.

Idea : If we had already such $\mathcal{L}$, then $\mathcal{L}|_{X \setminus \mathrm{supp}(D)}$ would have a holomorphic section without zeros $s|_{X \setminus \mathrm{supp}(D)}$. This would give a trivialization of $\mathcal{L}$ over $X \setminus \mathrm{supp}(D)$ where the section s is seen as the function 1. Near x such that $n_x \neq 0$, we cannot do exactly the same, but we can choose a coordinate z about x such that $z = 0$ is x and consider $z^{-n_x} s$ which would be a holomorphic section different from zero near x. This trivalizes $\mathcal{L}$ near x. In summary, we have the cases:

- s is seen as the function 1 in the trivialization we chose over $X \setminus \mathrm{supp}(D)$ and
- s is seen as z^{n_x} in the trivialization chosen near x.

To go from the trivialization over $X \setminus \mathrm{supp}(D)$ to the trivialization near x, it is enough to multiply by z^{n_x}. To go from a trivialization near x to a trivialization near y, it is enough to multiply by $z^{n_y - n_x}$ (in the case we have not chosen small enough neighborhoods).

More algebraic. A more "direct" construction of the line bundle directly defines its $\mathcal{O}_X$-module of local sections ($\mathcal{O}_X$ being the sheaf of holomorphic functions on X). This is the content of Exercise 11.