Exercise Sheet 5. A few questions

Exercice 1. Let $h \in \mathbb{C}(z)$. We write it as $h(z) = \alpha \prod_{i=1}^{k}(z-a_i)^{n_i}$ with $\alpha \in \mathbb{C}^*$, the $a_i \in \mathbb{C}$ pairwise distinct and $n_i \in \mathbb{Z}\setminus\{0\}$. Suppose that at least one of n_i is odd.

1. Show that $w^2 - h(z) \in \mathcal{M}(\mathbb{C}P^1)[w]$ is irreducible.
2. Show that the Riemann surface associated to $w^2 - h(z) \in \mathcal{M}(\mathbb{C}P^1)[w]$ is isomorphic (covering isomorphism) to the one associated to $w^2 - g(z)$ for some polynomial $g \in \mathbb{C}[z]$ with only simple roots.
3. Show that a compact and connected Riemann surface that can be written as a two-sheeted covering over $\mathbb{C}P^1$ is hyperelliptic **in the sense defined in TD 3**.
4. Consider ℓ distinct points $p_1, \dots, p_\ell \in \mathbb{C}P^1$. If ℓ is even, show that there exists a unique two-sheeted branched covering (modulo isomorphisms) such that $p_1, \dots, p_\ell$ are the ramification points. If ℓ is odd, show that there is no such a covering.

Exercice 2. Let $P \in \mathbb{C}[z, w, t]$ be an irreducible homogeneous polynomial with $\deg P \geq 2$ and consider the set $X = \{[z, w, t] \in \mathbb{C}P^2 : P(z, w, t) = 0\}$. Show that X is a Riemann surface if and only if $\{(z, w, t) \in \mathbb{C}^3 : \partial P/\partial z(z, w, t) = \partial P/\partial w(z, w, t) = \partial P/\partial t(z, w, t) = 0\} = \{(0, 0, 0)\}$.

In what follows, S **is a compact and connected Riemann surface**.

Exercice 3. Suppose that the genus of S is 2 and let $\pi_1, \pi_2 : S \to \mathbb{C}P^1$ be two two-sheeted coverings. Show that there exists a biholomorphism $F : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that $F \circ \pi_1 = \pi_2$. Deduce that the branching points (points of S) of π_1 and of π_2 are the same.

Exercice 4. Let X be a complex manifold and $Y \subset X$ be a closed complex submanifold of X. Show that, for every holomorphic map $f : S \to X$,

$$f^{-1}(Y) = S \quad \text{or} \quad f^{-1}(Y) \text{ is finite.}$$

Exercice 5. Let f be meromorphic on S and $\mathcal{L} : S \to [-\infty, \infty]$ be defined by $\mathcal{L}(z) = \log|f(z)|$.

1. Show that $\mathcal{L} \in L^1(S)$ in the sense[1] where, for each smooth 2-form ρ, we have $\mathcal{L} \in L^1(S, \rho)$.
2. If $\operatorname{div}(f) = \sum_{i=1}^{n} n_i[p_i]$, show that $\Delta\mathcal{L} = 2\pi \sum_{i=1}^{n} n_i \delta_{p_i}$ in the sense where

$$\int_S \mathcal{L}\Delta h = 2\pi \sum_{i=1}^{n} n_i h(p_i) \quad \text{for every } h \in C^\infty(S).$$

Let us define the notion of holomorphic fiber bundle with fiber $\mathbb{C}P^1$. Take a complex manifold E and a holomorphic map $\pi : E \to S$. We will say that (E, π) is a holomorphic fiber bundle of fiber $\mathbb{C}P^1$ if for every $p \in S$ there exists an open neighborhood U of p and a biholomorphism $\varphi : U \times \mathbb{C}P^1 \to \pi^{-1}(U)$ satisfying $\pi \circ \varphi = \pi_1$, where $\pi_1 : U \times \mathbb{C}P^1$ is the projection onto the first coordinate. A section will be a holomorphic map $f : S \to E$ such that $\pi \circ f = \mathrm{Id}_S$.

Exercice 6. Define a holomorphic fiber bundle E over S of fiber $\mathbb{C}P^1$ endowed with a holomorphic section σ of E such that there is a "natural" identification between the *meromorphic* 1-*forms over* S and the *holomorphic sections of* E *different from* σ.

[1] Notice that it is enough to verify this for one nowhere zero 2-form.