Exercise Sheet 4. Coverings, 1-forms, algebraic geometry ...

Here X and Y are two compact and connected Riemann surfaces.

Exercise 1 (Still about coverings)**.**

1. Do there exist non-ramified coverings of $\mathbb{C}P^1$?
2. What about ramified coverings of $\mathbb{C}P^1$ with only one ramification point?
3. Show that every non-constant holomorphic map between two elliptic curves is non-ramified.
4. Show that every non-constant holomorphic function between two surfaces of the same genus $g \geq 2$ is a biholomorphism.
5. Show that, if there is a non-constant holomorphic function $f : X \to Y$, then the genus of X is greater or equal than the genus of Y.

Let $p : X \to Y$ be a non-zero holomorphic map. For a divisor $D = \sum n_x[x] \in \mathrm{Div}(X)$ we denote p_*D the *image divisor*[1] $p_*D = \sum n_x[p(x)]$. On the other hand, for $D = \sum n_y[y] \in \mathrm{Div}(Y)$ and p non-constant, we denote p^*D the *pullback divisor* $p^*D = \sum (\mathrm{mult}_x p) n_{p(x)}[x]$.

Exercise 2 (Riemann-Hurwitz)**.** Let $p : X \to Y$ be holomorphic and non-constant.

1. For a non-zero meromorphic function $f : Y \dashrightarrow \mathbb{C}$, show that $\mathrm{div}(p^*f) = p^*\mathrm{div}(f)$.
2. For a 1-form ω not identically zero over Y show that $\mathrm{div}(p^*\omega) = p^*\mathrm{div}(\omega) + R_p$, where $R_p \in \mathrm{Div}(X)$ is the ramification divisor of p.
3. For a 1-form ω not identically zero over Y, show that $p_*(\mathrm{div}(p^*\omega)) = n_p\mathrm{div}(\omega) + p_*(R_p)$, where n_p is the degree of p.

Exercise 3. Let $f : X \to \mathbb{C}P^1$ be a non-constant holomorphic function and consider a point $z \in \mathbb{C}P^1$ which is not a ramification point. Show that, if $g : X \to \mathbb{C}P^1$ is a holomorphic function that separates the points of $f^{-1}(z)$, then $\mathcal{M}(X) = \mathbb{C}(f, g)$.

Exercise 4. Let $k \geq 1$ and $P = (z - a_1) \dots (z - a_k)$ be a monic polynomial with simple roots. We have seen that this defines a ramified covering $\pi : \sqrt{P(z)} \to \mathbb{C}P^1$ and that we have a biholomorphism between $\pi^{-1}(\mathbb{C})$ and $S := \{(z, w) \in \mathbb{C}^2 : w^2 = P(z)\}$. Let us restrict the 1-form dz defined over $\mathbb{C}^2$ to S and transport it to $\pi^{-1}(\mathbb{C})$ by keeping the notation dz.

1. Show that dz may be extended to a meromorphic 1-form over $\sqrt{P(z)}$.
2. Find $\mathrm{div}(dz)$.
3. If $k \geq 3$, find a non-zero meromorphic function $f : \sqrt{P(z)} \dashrightarrow \mathbb{C}$ whose divisor satisfies $\mathrm{div}(f) + \mathrm{div}(dz) \geq 0$. Can we find such a function if $k \leq 2$?
4. Find $\lfloor \frac{k-1}{2} \rfloor$ holomorphic 1-forms that are linearly independent.

[1] I.e., $p_* : \mathrm{Div}(X) \to \mathrm{Div}(Y)$ is the morphism induced by p when we think $\mathrm{Div}(X)$ and $\mathrm{Div}(Y)$ as free abelian groups generated by X and Y.

Exercise 5. Let $U \subset X$ be a cofinite open set and denote by $\mathcal{O}(U)$ the ring of meromorphic functions over X that are holomorphic over U.

1. Show that, if $U \subsetneq \mathbb{C}P^1$, then $\mathcal{O}(U)$ is a unique factorization domain.

2. Show that $\mathcal{O}(\mathcal{E} \setminus \{p\})$ is not a unique factorization domain if $\mathcal{E}$ is an elliptic curve and $p \in \mathcal{E}$.

Exercise 6. Let $X \subset \mathbb{C}P^2$ be a Riemann surface obtained as the zeros of some homogeneous irreducible polynomial $P \in \mathbb{C}[z, w, t]$. Show that

$$\deg(X) = \deg(P).$$

Exercise 7. Let $X \subset \mathbb{C}P^2$ be a Riemann surface obtained as the zeros of a homogeneous irreducible polynomial $P \in \mathbb{C}[z, w, t]$.

1. Let $f, g \in \mathbb{C}[z, w, t]$ be two homogeneous polynomials of the same degree such that $P \nmid g$. Show that f/g defines a meromorphic function over X.

2. Show that every holomorphic function over X can be constructed in this way.

In the following exercise, we can accept without proof (we will see it later in the course) the next theorem. *For every $x \in X$, there exists a non-constant meromorphic function that is holomorphic outside of x.*

Exercise 8. Let us consider the category $\mathcal{S}$ of Riemann surfaces where the morphisms are the holomorphic non-constant maps and the category $\mathcal{E}$ of finitely generated field extensions of transcendence degree[2] 1 of $\mathbb{C}$ where the morphisms are the unital ring morphisms that fix $\mathbb{C}$. Let us take the contravariant function $F : \mathcal{S} \to \mathcal{E}$ given by

- $F(X) = \mathcal{M}(X)$ the field of meromorphic functions over X and
- for every $\varphi : X \to Y$ the morphism $F(\varphi) : \mathcal{M}(Y) \to \mathcal{M}(X)$ is given by $f \mapsto f \circ \varphi$.

Show that F is an equivalence of categories.

[2]In other words, the objects of $\mathcal{E}$ are the finite extensions K of $\mathbb{C}(t) \simeq \mathcal{M}(\mathbb{C}P^1)$, but we forget how $\mathbb{C}(t)$ is embedded in K and retain only how $\mathbb{C}$ is embedded in K.