Exercise Sheet 3. Covering spaces and algebraic functions

Exercise 1. Let X and Y be two compact and connected Riemann surfaces and let $f : X \to Y$ be a holomorphic non-constant map.

1. Show that every continuous map $T : X \to X$ such that $f \circ T = f$ is holomorphic.
2. Denote by $S \subset Y$ the set of ramification points and $\tilde{X} = X \setminus f^{-1}(S)$. Show that every biholomorphism $\tilde{T} : \tilde{X} \to \tilde{X}$ that satisfies $f|_{\tilde{X}} \circ \tilde{T} = f|_{\tilde{X}}$ can be extended to a biholomorphism $T : X \to X$ satisfying $f \circ T = f$.
3. Let $T : X \to X$ be a biholomorphism that satisfies $f \circ T = f$ and fix $x \in X$. Show that if $x \in X$ is a branching point of f of order n then $T(x)$ is a branching point of f of order n.

Exercise 2. Let $\Gamma_1, \Gamma_2 \subset \mathbb{C}$ be two subgroups isomorphic to $\mathbb{Z}^2$.

1. Show that, for every non-constant holomorphic map $f : \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Gamma_2$ satisfying $f(0) = 0$, there exists a unique $\alpha \in \mathbb{C}^*$ such that $\alpha\Gamma_1 \subset \Gamma_2$ and satisfying $p_2(\alpha z) = f(p_1(z))$ for every $z \in \mathbb{C}$, where $p_i : \mathbb{C} \to \mathbb{C}/\Gamma_i$ is the canonical projection.
2. Let $H : [0,1] \times \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Gamma_2$ be a continuous map such that $H(t, \cdot) : \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Gamma_2$ is holomorphic for every $t \in [0,1]$. Show the existence of a continuous $\gamma : [0,1] \to \mathbb{C}/\Gamma_2$ satisfying $H(t,p) = H(0,p) + \gamma(t)$ for all $(t,p) \in [0,1] \times \mathbb{C}/\Gamma_1$. Conclude that the connected component of the identity in the group of automorphisms of an elliptic curve only contains translations.

Exercise 3 (Some examples)**.**

1. Find the ramification points of the map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$
$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$
What are the biholomorphisms $T : \mathbb{C}P^1 \to \mathbb{C}P^1$ satisfying $f \circ T = f$?
2. Let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be defined by a non-constant polynomial and let $T : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a biholomorphism such that $f \circ T = f$.
   — Show that T is given by $T(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ and α a root of unity.
   — Show that if $\alpha = 1$ then $\beta = 0$.
   — Give an example of f and T satisfying $\beta \neq 0$.
   — Give an example of f of degree 3 such that the only possible T is the identity.
3. Let $\Gamma_1, \Gamma_2 \subset \mathbb{C}$ be two subgroups isomorphic to $\mathbb{Z}^2$ satisfying $\Gamma_1 \subset \Gamma_2$. Show that the canonical map $p : \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Gamma_2$ is a holomorphic non-ramified covering map and find the group $\mathrm{Aut}(\pi)$ of automorphisms T of $\mathbb{C}/\Gamma_1$ that satisfy $p \circ T = p$.

Exercise 4. Let $\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

1. Let $n > 0$. Show that $g : \mathbb{D}^* \to \mathbb{D}^*$, defined by $g(z) = z^n$, is an unramified covering map.

2. Let X be a (connected) Riemann surface and let $f : X \to \mathbb{D}^*$ be a holomorphic (unramified) covering map of finite degree. Show that there exists a biholomorphism $\varphi : X \to \mathbb{D}^*$ and an integer $n > 0$ such that, for every $x \in X$, $f(x) = \varphi(x)^n$.
3. Let X be a *not necessarily connected* Riemann surface and let $f : X \to \mathbb{D}^*$ be a holomorphic (unramified) covering map of finite degree. Show that there exists an integer $k > 0$, integers $n_1, \dots, n_k > 0$ and a biholomorphism $\varphi : X \to \mathbb{D}_1^* \sqcup \cdots \sqcup \mathbb{D}_k^*$ such that, for every $x \in \varphi^{-1}(\mathbb{D}_i^*)$, $f(x) = \varphi(x)^{n_i}$. Here the $\mathbb{D}_i^*$ are disjoint copies of $\mathbb{D}^*$ and $z \mapsto z^{n_i}$ goes from $\mathbb{D}_i^*$ to $\mathbb{D}^*$.

Hyperelliptic curves over $\mathbb{C}$

Let us fix a monic polynomial $P \in \mathbb{C}[z]$ of pairwise distinct roots, $P(z) = (z - a_1) \dots (z - a_k)$ with $k \geq 1$. Consider the set

$$S = \{(z, w) \in \mathbb{C}^2 : w^2 = P(z)\}.$$

We have already seen in the exercise sheet of TD1 that the closure of S in $\mathbb{C}P^2$ is not a Riemann surface if $k \geq 4$. Let us build a "better" compactification of S. Think of P as a meromorphic function on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ and take the Riemann surface associated to $w^2 - P \in \mathcal{M}(\mathbb{C}P^1)[w]$ that we will call $\sqrt{P(z)}$ and that comes equipped with a projection $p : \sqrt{P(z)} \to \mathbb{C}P^1$.

Exercise 5. Construct a holomorphic map $\Phi : S \to \sqrt{P(z)}$ such that $p(\Phi(z, w)) = z$ for every $z \in \mathbb{C}$ and show that it is a biholomorphism over its image.

Take $\bar{S} =$ closure of S in $\mathbb{C}P^2$.

Exercise 6. Show that the map $\Phi^{-1} : \Phi(S) \to \mathbb{C}P^2$ can be extended to a holomorphic map $\Phi^{-1} : \sqrt{P(z)} \to \mathbb{C}P^2$ whose image is $\bar{S}$.

Exercise 7. Find the genus of $\sqrt{P(z)}$.

A Riemann surface obtained as $\sqrt{P(z)}$ is called a *hyperelliptic curve*.

Puiseux's theorem

The goal of the following exercise is to describe the algebraic closure of the field $\mathbb{C}\{\{z\}\}$ of germs of meromorphic functions at 0. More precisely, $\mathbb{C}\{\{z\}\}$ is formed by the formal Laurent series with a positive convergence radius. For each $m \geq 1$, take $\mathbb{C}\{\{z^{1/m}\}\}$ a copy of $\mathbb{C}\{\{z\}\}$, where the new variable is called $z^{1/m}$. For m, n satisfying $m|n$ we have the canonical inclusion $i_{n,m} : \mathbb{C}\{\{z^{1/m}\}\} \to \mathbb{C}\{\{z^{1/n}\}\}$ given by

$$i_{n,m}\left(\sum_{\ell=k}^{\infty} a_\ell (z^{1/m})^\ell\right) = \sum_{\ell=k}^{\infty} a_\ell (z^{1/n})^{\ell n/m}.$$

This gives us an inductive system $((\mathbb{C}\{\{z^{1/m}\}\})_{m\geq 1}, (i_{n,m})_{m|n})$ and we may consider its colimit (direct limit) $\mathrm{Pui}(z)$. We will prove that

$$\mathrm{Pui}(z) \text{ is the algebraic closure of } \mathbb{C}\{\{z\}\}.$$

Exercise 8. Let $F(z, w) = w^n + a_1(z) w^{n-1} + \cdots + a_n(z) \in \mathbb{C}\{\{z\}\}[w]$ be an irreducible polynomial. Show that there exists an element $\varphi \in \mathbb{C}\{\{\zeta\}\}$ that satisfies

$$F(\zeta^n, \varphi(\zeta)) = 0 \in \mathbb{C}\{\{\zeta\}\}.$$